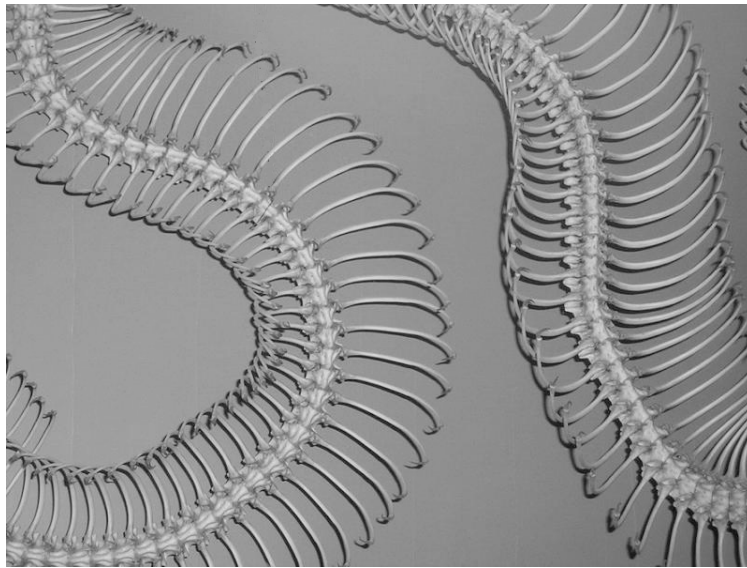


Snakes, Swan's Necks, and Puppy Dog Tails

It has been noted in passing that seven cervical vertebrae in mammalian necks seems to be a threshold of sorts for movements (refs). If you add an additional link, the movements take on a looser quality that does not seem quite 'neck-like' and stiffening one linkage, as might happen with a fusion or arthritic restrictions, leads to a substantial reduction in movement. One might speculate that the number of vertebrae and the amount of movement in the intervertebral linkages might be in some way set by evolution to achieve the needs of a mammalian neck (ref). With a very small number of exceptions (manatees, and two types of sloth), all mammals have 7 cervical vertebrae.



A snake's skeleton has a series of nearly identical elements composed of a vertebra and two attached ribs. The concatenation of many small movements between successive vertebrae allows the snake to take a wide variety of sinuous shapes.

That leads to curiosity about the implications of chains of many similar linkages, such as occur in a cat's tail, a snake's body, or a swan's neck. A swan's neck may have up to 25 cervical

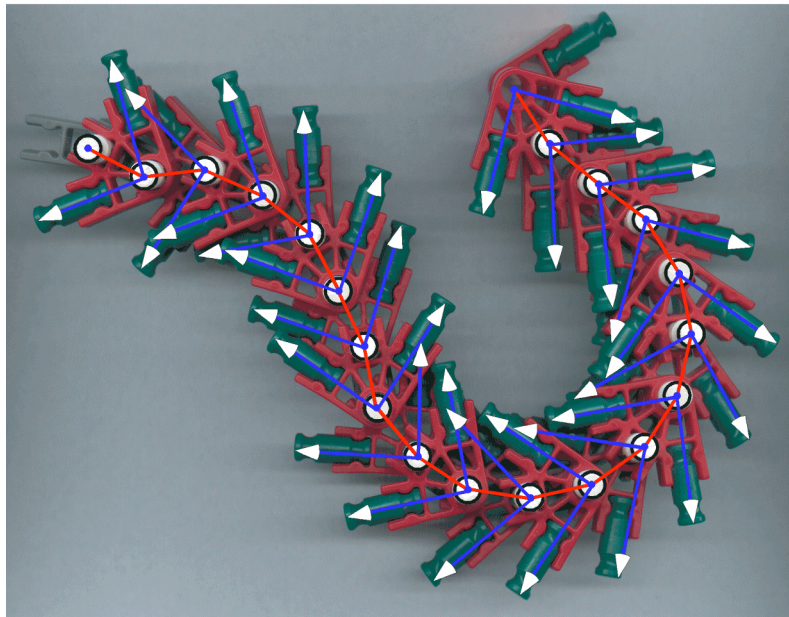
vertebrae. Some of the aquatic reptiles of the Mesozoic, plesiosaurs, had up to 40 cervical vertebrae. The cat's tail may have 21-23 caudal vertebrae. Snakes have largely identical series of vertebral segments, with attached ribs. All of these anatomical structures have a great deal of mobility and a similar manner of moving.

In this chapter, we will consider the dynamics of a mechanical system that has a high degree of repetition of concatenated identical elements. While we start with the idea of a snake or a swan's neck, we will be concerned with an abstract entity that we will call an artificial snake. The results may not be directly applicable to actual snakes or swans necks or puppy dog tails, because a close examination will reveal that the elements in those structures are not in fact identical. Snake spines come closest. Bird necks are most differentiated.

Artificial Snakes

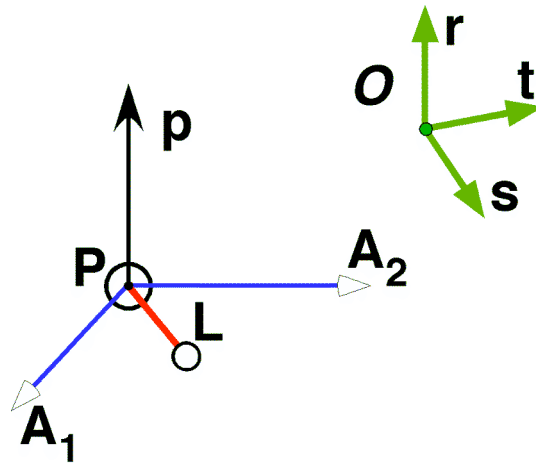
We will talk about joints and muscles and how muscles move bones at joints. While we start with the concepts of actual joints and muscles, we will again recast those entities into abstractions and consider the formal properties of groups of muscles and joints working in concert.

Toy Snake



An artificial snake is created by linking a series of identical elements that can move about a central pivot point. This mechanism can be described abstractly by set of linked framed vectors.

We will start with a very simple toy. A toy snake is constructed by concatenating a number of identical elements $[E_1, E_2, E_3, \dots, E_n]$. Each has a pivot point, which is taken to be the location $[P]$ of the element. In the illustrated instance all the pivot points have an axis of rotation $[p]$ that is directed perpendicular to the reference plane $[p = r]$ and the pivot points are linked to each other in directions that extend parallel to the reference plane. The linkage $[L]$ is an extension of the element to the next pivot point and it defines a direction relative to the element $[s]$. The direction of the link and the direction of the pivot axis define an orientation frame $[O]$. We will assume that the mutually perpendicular direction that your thumb points when the fingers of your right hand curl from the pivot axis to the linkage direction will be the final axis of the orientation frame $[t]$.



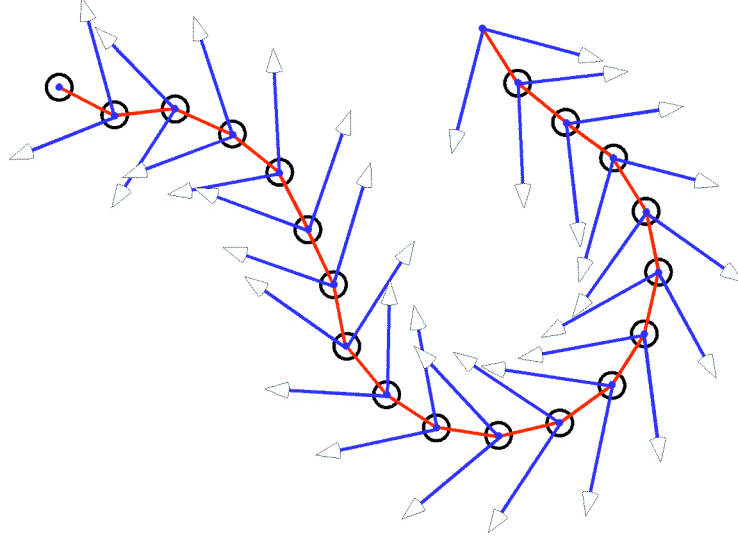
The frame vectors of a single element

In addition, we will have two armatures that extend away from the pivot point in the plane of the pivot axis, that is, parallel with the reference plane. The two armatures $[A_1, A_2]$ will extend symmetrically to either side of the linkage so as to form a right angle between them. Their directions will be a_1 and a_2 . Their length will be ' α '. Therefore, each element of the snake may be written as a framed vector.

$$E_n = \{P_n, L_n, A_{1n}, A_{2n}, \{r_n, s_n, t_n\}\};$$

$$L_n = \lambda s_n,$$

$$A_{1n} = \alpha a_{1n} \text{ and } A_{2n} = \alpha a_{2n}.$$



The abstract toy snake is expressed by a series of linked elements composed of a pivot point at a particular location and two extended vectors that arise from the pivot point and extend symmetrically to either side of the axis of the element. The element is orientable. Each element can be described by a framed vector.

The rotation between elements is given by the quaternion

$$\mathbf{R}_n = \mathbf{R}_n(\theta, \mathbf{p}) = \cos \theta_n + \mathbf{p} \sin \theta_n = \cos \theta_n + \mathbf{r} \sin \theta_n .$$

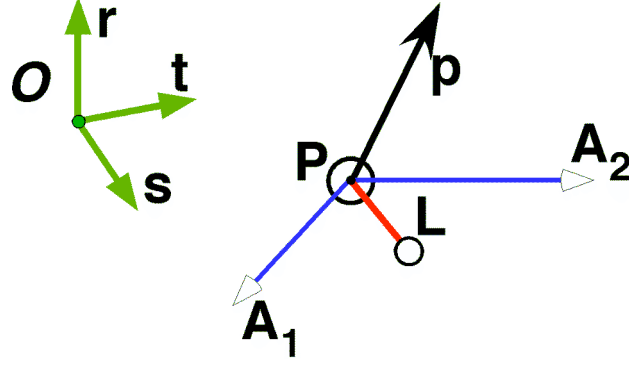
We can write down the expression for the snake from these expressions.

$$\begin{aligned} \mathbf{E}_{n+1} &= \{ \mathbf{P}_n + \mathbf{L}_n, \mathbf{L}_{n+1}, \mathbf{A}_{1n+1}, \mathbf{A}_{2n+1}, \{ \mathbf{r}_n, \mathbf{R}_{n+1} * \mathbf{s}_n, \mathbf{R}_{n+1} * \mathbf{t}_n \} \}; \\ \mathbf{L}_{n+1} &= \mathbf{R}_{n+1} * \lambda \mathbf{s}_n , \\ \mathbf{A}_{1n+1} &= \mathbf{R}_{n+1} * \alpha \mathbf{a}_{1n} \text{ and } \mathbf{A}_{2n+1} = \mathbf{R}_{n+1} * \alpha \mathbf{a}_{2n} . \end{aligned}$$

We can place it in space by specifying the location and orientation of the first element.

$$\begin{aligned} \mathbf{E}_1 &= \{ \mathbf{P}_1, \mathbf{L}_1, \mathbf{A}_{11}, \mathbf{A}_{21}, \{ \mathbf{r}_1, \mathbf{s}_1, \mathbf{t}_1 \} \}; \\ \mathbf{L}_1 &= \lambda \mathbf{s}_1 , \\ \mathbf{A}_{11} &= \alpha \mathbf{a}_{11} \text{ and } \mathbf{A}_{21} = \alpha \mathbf{a}_{21} , \\ \mathbf{a}_{11} &= \mathbf{R} \left(-\frac{\pi}{4}, \mathbf{r} \right) * \mathbf{s} \text{ and } \mathbf{a}_{21} = \mathbf{R} \left(+\frac{\pi}{4}, \mathbf{r} \right) * \mathbf{s} . \end{aligned}$$

Since the axis of the pivot joint is fixed relative to the element and it is aligned with the \mathbf{r} axis, the calculation is very straight-forward. We can also use the full angle form of quaternion multiplication to compute the new values of the frame elements.



The pivot axis is allowed to take any direction relative to the element. Since the pivot axis is no longer perpendicular to the plane of the element, the \mathbf{r} vector of the frame may be defined as the direction of the ratio of two of the extension vectors.

Having carried the analysis to this point in an essentially two-dimensional situation, it is easy to write down the description for an arbitrary snake in three dimensions. The main difference is that the pivot axis can take on any direction, therefore, we need to use the half angle expression for rotation.

$$\begin{aligned}
 \mathbf{E}_{n+1} &= \{ \mathbf{P}_{n+1}, \mathbf{L}_{n+1}, \mathbf{A}_{1n+1}, \mathbf{A}_{2n+1}, \{ \mathbf{r}_{n+1}, \mathbf{s}_{n+1}, \mathbf{t}_{n+1} \} \} \\
 &= \{ \mathbf{P}_n + \mathbf{L}_n, \mathbf{L}_{n+1}, \mathbf{A}_{1n+1}, \mathbf{A}_{2n+1}, \{ \rho_{n+1} * \mathbf{r}_n * \rho_{n+1}^{-1}, \rho_{n+1} * \mathbf{s}_n * \rho_{n+1}^{-1}, \rho_{n+1} * \mathbf{t}_n * \rho_{n+1}^{-1} \} \}; \\
 \rho_{n+1} &= \cos \frac{\theta}{2} + \mathbf{p}_{n+1} \sin \frac{\theta}{2}, \\
 \mathbf{L}_{n+1} &= \rho_{n+1} * \lambda \mathbf{s}_n * \rho_{n+1}^{-1}, \\
 \mathbf{A}_{1n+1} &= \rho_{n+1} * \alpha \mathbf{a}_{1n} * \rho_{n+1}^{-1} \text{ and } \mathbf{A}_{2n+1} = \rho_{n+1} * \alpha \mathbf{a}_{2n} * \rho_{n+1}^{-1}, \\
 \mathbf{a}_{1n+1} &= \mathbf{R} \left(-\frac{\pi}{4}, \mathbf{r}_{n+1} \right) * \mathbf{s} \text{ and } \mathbf{a}_{2n+1} = \mathbf{R} \left(+\frac{\pi}{4}, \mathbf{r}_{n+1} \right) * \mathbf{s}, \\
 \mathbf{a}_{1n+1} &= \frac{\mathbf{A}_{1n+1}}{|\mathbf{A}_{1n+1}|}, \\
 \mathbf{r} &= \mathbf{UV} \left[\frac{\mathbf{A}_{2n}}{\mathbf{A}_{1n}} \right] = \mathbf{UV} \left[\frac{\mathbf{A}_{2n}}{\mathbf{L}} \right] = \mathbf{UV} \left[\frac{\mathbf{L}}{\mathbf{A}_{1n}} \right].
 \end{aligned}$$

More generally, create a real or hypothetical vertical or lateral extension that is moved with the element.

The extension elements may be written as functions of the orientation frame of reference. Then, we need only keep track of the locations and orientations.