ON QUATERNIONS.

LECTURE I.

Gentlemen,

In the preceding Lectures of the present Term, we have taken a rapid view of the chief facts and laws of Astronomy, its leading principles and methods and results. After some general and preliminary remarks on the connexion between metaphysical and physical science, we have seen how the observation of the elementary phenomena of the Heavens may be assisted, and rendered more precise, by means of astronomical instruments, accompanied with astronomical reductions. An outline of Uranography has been given; the laws of Kepler for the Solar System have been stated and illustrated; with the inductive evidence from facts by which their truth may be established. It has been shewn that these laws extend, not only to the Planets known in Kepler's time, namely, Mercury, Venus, Mars, Jupiter, and Saturn, with which our Earth must be enumerated, but also to the various other planets since detected: to Uranus, to Ceres, Pallas, Juno, and Vesta; and to those others of more recent date, in the order of human knowledge, of which no fewer than six have been found within the last two years and a half; to Astræa, Neptune, Hebe, Iris, Flora, and Metis: among which Neptune is remarkable, as having had its existence foreshewn by mathematical calculation, and Metis is interesting to us Irishmen, as having been discovered at an Irish observatory. It has also been shewn you that these celebrated laws of Kepler are themselves mathematically included in one still greater Law, with which the name of Newton is associated; and that thus, as Newton himself demonstrated, in his immortal work, the Principia, the rules of the elliptic motion of the planets are consequences of the principle of universal Gravitation, proportional directly to the mass, and inversely to the square of the distance. With the help of this great principle, or law, of Newton's, combined with proper observations and experiments,—especially, with the Cavendish experiment, as lately repeated by Baily,—not only have the *shape* and *size* of the earth which we inhabit, but even (as you have seen explained and illustrated) its very *weight* has been determined; the number of millions of millions of millions of tons of matter, which this vast globe contains, has been (approximately) assigned. And not only have the motions of that Earth of our's around and with its own axis, and round the sun, been established, but that great central body of our system, the Sun, through the persevering application of those faculties which God has given to man, has itself (as you have likewise seen) been measured and weighed, with the line and balance of science.

2. Such having been our joint contemplations in this place, before the adjournment of these discourses on account of the Examinations for Fellowships, you may remember that it was announced that at our re-assembling we should proceed to the consideration of a certain new mathematical Method, or Calculus, which has for some years past occupied a large share of my own attention, but which I have hitherto abstained from introducing, except by allusion, to the notice of those who have honoured here my lectures with their attendance. I refer, as you are aware, to what I have called the CALCULUS of QUATERNIONS, and have applied to the solution of many geometrical and physical problems. However much this new calculus, or method, may naturally have interested myself, there has existed, in my mind, until the present time, a fear of seeming egotistical, if I should offer to the attention of my hearers in this University an account of such investigations or speculations of my own. Accordingly, with the exception of a short sketch, in the year 1845, of the application to spherical trigonometry of those fundamental conceptions and expressions respecting Quaternions, which I had been led to form in 1843, and had in the last mentioned year communicated to the Royal Irish Academy, I have abstained

from entering on the subject in former courses of Lectures:—unless it be regarded as an exception to this rule, that in the extraordinary or supplementary Course which I delivered here, in the winter of 1846, on the occasion of the theoretical discovery of the distant planet Neptune, I ventured to introduce that theory of *Hodographs*, which, in the regular course for 1847, I afterwards more fully developed; and which had been suggested to me as a geometrical interpretation, or construction, of some integrations of equations in physical astronomy whereto I had been conducted by the Method of Quaternions. But since, on the one hand, it has of late been formally announced (as it is stated to me) that the Professor of Mathematics in this University intends to lecture on that Method of mine in the winter of the present year, in connexion, probably, with some of his own original researches; and to make it, or a part of it, one of the subjects of his public Examination of the Candidates for Fellowship in the summer of 1849; while, on the other hand, the theory itself has been acquiring, under my own continued study, a wider extension, and perhaps also a firmer consistency: it seems to myself,—and by some mathematical friends, among whom the Professor just referred to is included, I am encouraged to think that it is their opinion too,—that the time has arrived, when instead of its being an obtrusion for me to state here, in the execution of my own professorial office, my views respecting Quaternions, it would, on the contrary, be rather a dereliction of my duty, or a blameable remissness therein, if I were longer to omit to state those views in this place, at least by sketch and outline.

3. And inasmuch as I am not aware that any one has hitherto professed to detect error in any of those geometrical and physical *theorems* to which the Method has conducted me; while yet I cannot but perceive it to be the feeling of several persons, among my mathematical friends and acquaintances, that in the existing state of the published expositions of my own views upon the subject, some degree of obscurity still hangs over its logical and metaphysical *principles:* so that the admitted correctness of the *results* of this new Calculus may appear, even to candid and not unfriendly lookers-on, to be, in some sense, *accidental*, rather than necessary, so far as the conceptions and reasonings have

hitherto been formally set forth by me: it therefore seems to be, upon the whole, the most expedient plan which can be adopted on the present occasion, that I should state, as distinctly and as fully as my own limited powers of expression, and as your remaining time in this Course will allow, the *fontal thoughts*, the *primal views*, the *initial attitudes of mind*, from which the others flow, and to which they are subordinated. And if, in the fulfilment of this purpose, the adoption of a somewhat *metaphysical style* of expression on some fundamental points may be at least forgiven me, as inevitable, still more may I look to be excused, if not approved of, should I take, even by preference, my *illustrations from Astronomy*, in this Supplementary Course of Lectures, which, as you know, arises out of, and is appended to a Course more strictly and properly astronomical.

4. The object which I shall propose to myself, in the Lecture of this day, is the statement of the significations, at least the *primary* significations, which I attach, in the Calculus of Quaternions, to the four following familiar marks of combination of symbols,

$$+ \quad - \quad \times \quad \div$$

which marks, or signs, are universally known to correspond, in arithmetic and in ordinary algebra, to the four *operations* known by the names of Addition, Subtraction, Multiplication, and Division. The *new* significations of these four signs have a sufficient *analogy* to the *old* ones, to make me think it convenient to *retain the signs* themselves; and yet a sufficient *distinction* exists, to render a *preliminary comment* not superfluous: or rather it is *indispensable* that as clear a definition, or at least *exposition*, of the precise force of each of these old marks, used in new senses, should be given, as it is in my power to give. Perhaps, indeed, I may not find it possible, to-day, to speak with what may seem the requisite degree of *fulness* of such exposition, of more than the *two first* of these four signs; although I hope to touch upon the two last of them also.

5. First, then, I wish to be allowed to say, in *general* terms (though conscious that they will need to be afterwards particularized), that I regard the two connected but contrasted marks or signs,

$$+ \text{ and } -,$$

as being respectively and *primarily characteristics of the* SYNTHESIS *and* ANALYSIS *of a* STATE *of a Progression*, according as this state is considered as being *derived from*, or *compared with*, some *other state* of that progression. And, with the same kind of generality of expression, I may observe here that I regard in like manner the *other pair* of connected and contrasted marks already mentioned, namely,

$$\times \text{ and } \div,$$

(when taken in what I look upon as their respectively *primary* significations), as being signs or characteristics of the corresponding SYNTHESIS and ANALYSIS of a STEP, in any such progression of states, according as that *step* is considered as *derived from*, or *compared with*, some *other step* in the same progression. But I am aware that this very general and preliminary statement cannot fail to appear vague, and that it is likely to seem also obscure, until it is rendered precise and clear by examples and illustrations, which the plan of these Lectures requires that I should select from Geometry, while it allows me to clothe them in an Astronomical garb. And I shall begin by endeavouring thus to illustrate and exemplify the view here taken of the sign −, which we may continue to *read*, as usual, MINUS, although the operation, of which it is now conceived to direct the performance, is not to be confounded with arithmetical, nor even, in *all* respects, with common algebraical subtraction.

6. I have said that I regard, *primarily*, this sign,

−, or Minus,

as the mark or characteristic of an *analysis of one state* of a progression, when considered as *compared with another* state of that progression. To illustrate this very general view, which has been here propounded, at first, under a metaphysical rather than a mathematical form, by proceeding to apply it under the limitations which the science of *geometry* suggests, let SPACE be now regarded as the *field* of the progression which is to be studied, and POINTS as the *states* of that progression. You will then see that in conformity with the general view already enunciated, and as its geometrical particularization, I am led to regard the word "Minus," or the mark −, in geometry, as the sign or characteristic of the analysis of one geometrical position (in space), as compared with another (such) position. The *comparison of one mathematical point with another*, with a view to the determination of what may be called their *ordinal relation*, or their *relative position* in space, is in fact the investigation of the GEOMETRICAL DIFFERENCE of the two points compared, in that *sole* respect, namely, *position*, in which two mathematical points *can differ* from each other. And even for this reason alone, although I think that other reasons will offer themselves to your own minds, when you shall be more familiar with this whole aspect of the matter, you might already grant it to be *not unnatural* to regard, as it has been stated that I *do* regard, this study or investigation of the relative position of two points in space, as being that *primary geometrical operation* which is *analogous to algebraic subtraction*, and which I propose accordingly to denote by the usual mark (−) of the well-known operation last mentioned. Without pretending, however, that I have yet exhibited sufficiently *conclusive* grounds for believing in the existence of such an *analogy*, I shall now proceed to illustrate, by *examples*, the modes of symbolical *expression* to which this belief, or view, conducts.

7. To illustrate first, by an astronomical example, the conception already mentioned, of the analysis of one geometrical position considered with reference to another, I shall here write down, as symbols for the two positions in space which are to be compared among themselves, the astronomical signs,

⊙ and ♁ ;

which represent or denote respectively the sun and earth, and are *here* supposed to signify, *not* the masses, nor the longitudes, of those two bodies, nor any other *quantities* or magnitudes connected with them, *but simply their* SITUATIONS, or the positions of their centres, regarded as mathematical POINTS in space. To make more manifest to the eye that these astronomical signs are here employed to denote points or positions alone, I shall write under each a *dot*, and under the dot a Roman capital letter, namely, A for the earth, and B for the sun, as follows:

⊙	♁	
·	·	(Fig. 1.)
B	A	

and shall suppose that the particular operation of what we have already called analysis, using that word in a very general and rather in a metaphysical than in a mathematical sense, which is now to be performed, consists in the proposed *investigation of the position of the sun*, B, *with respect to the earth*, A; the latter being regarded as comparatively simple and known; but the former as complex, or at least unknown and undetermined; and a relation being sought, which shall connect the one with the other. This conceived analytical operation is practically and astronomically performed, to *some* extent, whenever an observer, as for example, my assistant (or myself), at the Observatory of this University, with that great circular instrument of which you have a model here, directs a telescope to the sun: it is *completed*, for that particular time of observation, when, after all due micrometrical measurements and readings, after all reductions and calculations, founded in part on astronomical theory, and on facts previously determined, the same observer concludes and records the geocentric right ascension and declination, and (through the semidiameter) the radius vector (or distance) of the sun. In general, we are to conceive the required *analysis* of the position of the ANALYZAND POINT B, with respect to the ANALYZER POINT A, to be an operation such that, if it were *completely* performed, it would instruct us *not only* IN WHAT DIRECTION the point B is situated with respect to the point A; *but also*, AT WHAT DISTANCE from the latter the former point is placed. Regarded as a guide, or rule for going (if we *could* go) from one point to the other,—which RULE of transition would, however (according to the general and philosophical, rather than technically mathematical distinction between analysis and synthesis, on which this whole exposition is founded), be *itself* rather of a *synthetic* than of an *analytic* character,—the RESULT of this ordinal analysis might be supposed to tell us in the *first* place HOW WE SHOULD SET OUT: which conceived geometrical ACT, of *setting out in a suitable direction*, corresponds astronomically to the pointing, or *directing of the telescope*, in the *observation* just referred to. And the same synthetic rule, or the same result of a complete analysis, must then be supposed *also* to tell us, in the *second* place, HOW FAR WE OUGHT TO GO, in order to ARRIVE AT the sought point

B, after thus setting out from the given point A, in the proper direction of progress (this direction being, of course, here conceived to be *preserved* unaltered): which latter part of the supposed guidance or information corresponds to the astronomical inquiry, *how far off* is the sun, or other celestial object, at which we are now looking, with a telescope properly set?

8. Now the *whole sought* RESULT of this (conceived) complete analysis, of the position B with respect to the position A, whether it be regarded analytically as an *ordinal relation,* or synthetically as a *rule of transition,* is what I propose to *denote,* or signify, by the symbol

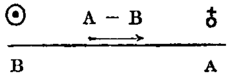
$$B - A,$$

formed by inserting the sign MINUS between the two separate symbols of the two points compared; the symbol of the *analyzand point* B being written to the *left* of the mark −, and the symbol of the *analyzer point* A being written to the *right* of the same mark; all which I design to illustrate by the following fuller diagram,

(Fig. 2.)

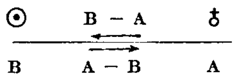
where the *arrow* indicates the *direction* in which it would be necessary to *set out* from the analyzer point, in order to *reach* the analyzand point; and a straight *line* is drawn to represent or picture the *progression,* of which those *points* are here conceived to be, respectively, the initial and final *states.* We may then, as often as we think proper, *paraphrase* (in this theory) the geometrical symbol B − A, by reading it aloud as follows, though it would be tedious always to do so: "B *analyzed with respect to* A, *as regards difference of geometrical position.*" But for common use it may be sufficient (as already noticed) to retain the shorter and more familiar mode of reading, "B minus A;" remembering, however, that (in the present theory) the DIFFERENCE thus *originally* or *primarily* indicated is one of POSITION, and *not of magnitude:* which, indeed, the *context* (so to speak) will always be sufficient to suggest, or to remind us of, whenever the symbols A and B are recognised as being what they are here supposed to be, namely, signs of *mathematical points.*

9. Had we chosen to *invert the order of the comparison,* or of the analysis of these two positions A and B, as related to each other, regarding the sun B as the given or known point, and the earth A as the sought or unknown one; we should have in *that* case done what in fact astronomers do in those investigations respecting the solar system, in which the motion of the earth as a planet about the sun, in obedience to Kepler's laws, is treated as an established general fact which it remains to argue from, and to develope into the particular consequences required for some particular question: whenever, in short, they seek rather the *heliocentric position of the earth,* than the *geocentric position of the sun;* and so propose to analyze what has been here called A with respect to B, rather than B with respect to A. And it would then have been proper, on the same general plan of notation, to have written the *opposite symbol* A − B, instead of the former symbol B − A; and also to have *inverted the arrow* in the diagram (because we now conceive ourselves as going rather from the sun to the earth, than from the earth to the sun); which diagram would thus assume the form,



(Fig. 3.)

Thus B − A and A − B are *symbols of two opposite* (or mutually inverse) *ordinal relations,* corresponding to two OPPOSITE STEPS or transitions in space, and mentally discovered, or brought into notice, by these *two opposite modes of analyzing the relative position of one common pair of mathematical points,* A and B; of which two opposite modes of ordinal analysis in space, with the two inverse relations thence resulting, the mutual connexion and contrast may be still more clearly perceived, if we bring them into one view by this diagram:



(Fig. 4.)

10. Using a *form of* WORDS, suggested by this mode of symbolical notation, I should not think it improper, and it would certainly be at least consistent with the manner in which the subject is here viewed, to say that

The Sun's ordinal relation to the Earth in space,
or, somewhat more concisely, that what is called in astronomy,
"The *Sun's Geocentric Position*" (including *distance*),
is *expressed by*, and is (in *that* sense) *equivalent*, or
(with the here proposed use of Minus) *symbolically equal to*

"The Sun's (absolute) Position in space,
MINUS the Earth's (absolute) Position."

And then, of course, we should be allowed, on the same plan, to say, conversely, that

"The Earth's Heliocentric Position" is equivalent or equal to
"The Earth's Position in space, *minus* the Sun's Position."

In the same new mode of speaking, the

"Position of Venus (in space), *minus* the Position of the Sun,"

would be a form of words equivalent to the usual phrase,

"Heliocentric Position of Venus."

And it is evident that examples of this sort might easily be multiplied.

11. According, then, to the view here taken of the word "Minus," or of the sign −, if employed, as we propose to employ it, in pure or applied *geometry*, this word or sign will denote *primarily* an *ordinal analysis in space;* or an *analysis* (or examination) *of the position of a mathematical point*, as *compared* with the position of *another* such point. And because, according to the foregoing illustrations, this sign or mark (Minus) directs us to DRAW, or to conceive as drawn, a *straight line connecting the two points*, which are proposed to be compared as to their relative positions, it might, perhaps, on this account be called the SIGN OF TRACTION. If we wish, however, to *diminish*, as far as possible, the *number of new terms*, we may call it still, as usual, the sign of SUBTRACTION; remembering only, that, in the view here proposed, there is NO *original* (nor necessary) *reference* whatever *to any subtraction of one* MAGNITUDE *from another*. Indeed, it is well known to every student of the elements of algebra that the word Minus, and the sign −, are, in those elements *also*, used very frequently to denote an operation which is

by no means identical with the taking away of a partial from a total magnitude, so as to find the *remaining part*: thus every algebraist is familiar with such results as these, that

(Negative Four) Minus (Positive Three) Equals (Negative Seven);

where, if mere magnitudes or quantities were attended to, and the adjectives "Positive and Negative" dropped, or neglected, and not replaced by any other equivalent words or marks, the resulting *number* "seven" would represent the (arithmetical) *sum*, and *not* the (arithmetical) *difference*, of the given numbers "four" and "three." And as, to prevent any risk of such confusion with a merely *arithmetical difference*, or with the result of a merely *arithmetical subtraction*, it is usual to speak of an *algebraical difference* and of *algebraical subtraction;* and thus to say, for example, that "Negative Seven" is the "algebraical difference" of "Negative Four" and "Positive Three;" or is obtained or obtainable by the "algebraical subtraction" of the latter from the former: so may (I think) that other and more *geometrical* sort of subtraction, which has been illustrated in this day's Lecture, be called, not inconveniently, for the sake of recognising a farther distinction or departure from the merely *popular* use of the word (subtraction), and on account of its connexion with a new and enlarged system of *symbols* in geometry, the SYMBOLICAL SUBTRACTION of A from B: and the resulting symbol of the ordinal relation of the latter point to the former, namely, the symbol B − A, may conveniently be called, in like manner, a SYMBOLICAL DIFFERENCE. It is in fact, as has been already remarked, in this new system of symbols, an expression for what may very naturally be called the *geometrical difference of the two points* B and A; that is to say, it is (in this system) a *symbol for the difference of the positions of those two mathematical points in space;* this difference being regarded as geometrically constructed, represented, or pictured, by the *straight line* drawn *from* A *to* B, which LINE is here *considered as having* (what it *has* in fact) *not only a determined length, but also a determined direction*, when the two *points*, A and B, themselves, are supposed to have two distinct and determined (or at least determinable) *positions*.

12. For my own part I cannot conceal that I hold it to be of great and even *fundamental* importance, to regard Pure Mathematics as being *primarily* the science of ORDER (in Time and Space), and *not* primarily the science of MAGNITUDE: if we would attain to a perfectly clear and thoroughly self-consistent view of this great and widely-stretching region, namely, the mathematical, of human thought and knowledge. In mathematical science the doctrine of magnitude, or of quantity, plays indeed a *very* important part, but *not*, as I conceive, the *most* important one. Its importance is SECONDARY and DERIVATIVE, *not primary* and *original*, according to the view which has long approved itself to my own mind, and in entertaining which I think that I could fortify myself by the sanction of some high authorities: although the opposite view is certainly more commonly received. If any one here should regard that opposite view, which refers all to magnitude, as the right one; and should find it impossible, or think it not worth the effort, to suspend even for a while the habit of such a reference, he may still give for a moment a geometrical interpretation to the symbol B − A, not *quite* inconsistent with that which has been above proposed, by regarding it as an *abbreviation* for this other symbol BO − AO, where AO and BO are *lines*, namely, the distances of the two points A and B from another point O, assumed on the same indefinite right line as those two points A, B, and lying beyond A with respect to B, or situate upon the line BA prolonged through A, as in this diagram:

(Fig. 5.)

Here the point O may be conceived, astronomically, to represent a superior planet, for example, Jupiter (♃), in opposition to the Sun (and in the Ecliptic); and it is evident that if we knew, for such a configuration, the distance AO in millions of miles, of the Earth from Jupiter, and also the greater distance BO of the Sun from the same superior planet at that time, we should only have to *subtract*, *arithmetically*, the former distance AO from the latter distance BO, for the purpose of finding the *distance* BO − AO, or BA, in millions of miles, between the earth and the sun; which distance, there might thus be some propriety or convenience, on *this* account, in denoting by the symbol B – A. That symbol, thus viewed, might even be conceived to suggest a reference to *direction* as well as distance; because the supposed line OA, prolonged through A, would in the figure tend to B; or, in astronomical language, the *jovicentric place of the Earth*, in the configuration supposed, would coincide, on the celestial sphere, with the *geocentric place of the Sun*. But I am far indeed from recommending to you to *complicate* the contemplation of the relative position of the two points A and B, at this early stage of the inquiry, by any reference of this sort to any *third* point O, thus foreign and arbitrarily assumed. On the contrary, I would *advise*, or even *request* you, for the present, to *abstain* from making, in your own minds, such a reference to any *foreign* point; and to accompany me, for some time longer, in considering *only* the INTERNAL relation of position of the TWO points, A and B, themselves: *agreeing* to regard this internal and ordinal relation of these two mathematical points in space (to whatever extent it may be found useful, or even necessary *hereafter*, to call in the aid of other points, or lines, or planes, for the purpose of more fully studying, and, above all, of *applying* that relation), as being sufficiently DENOTED, at *this* stage, by one or other of the two symbols, B – A or A – B, according as we choose to regard B or A as the analyzand point, and A or B as the analyzer.

13. I ask you then to concede to me, at least provisionally, and for a while, the privilege of employing this unusual mode of geometrical NOTATION, together with the new mode of geometrical INTERPRETATION above assigned to it: which modes, after all, do *not contradict* anything *previously established* in scientific language, nor lead to any real risk of confusion or of ambiguity, in geometrical science, by attaching any *new sense* to an *old sign:* since here the *sign itself* (B – A), as well as the signification, is new. The *component symbol* "minus" is indeed *old*, but it is used here in a *new connexion* with other elementary symbols; and the *new context*, hence arising, gives birth to a NEW COMPLEX SYMBOL, (B – A), in fixing the sense of which we may and must be guided by analogy, and general considerations:

old usages and received definitions failing to assign any determined signification to the new complex symbol thus produced. The interpretation which I propose does no more than *invest with sense*, through an *explanation* which is new, what had seemed before to be *devoid of sense*. It only *gives a meaning*, where none had been given before: namely, to a symbolical expression of the form "Point *minus* Point." This latter *form of words*, and the geometrical *notation* B − A to which it corresponds (A and B being still used as *signs of mathematical points*), had hitherto, according to the *received* and *usual* modes of geometrical interpretation, NO MEANING: but you will, perhaps, admit that these two connected forms of spoken and written expression were, *for that very reason*, only the *more free* to receive any *new* and *definitional sense:* especially one which you have seen to admit of beng *suggested* by so simple an *analogy to subtraction* as that which the *conception of difference* involves. It will, however, of course be necessary, for consistency, that we carefully *adhere* to such new interpretation, when it has once been by definition assigned: unless and until we find reasons (if such reasons shall ever be found) which may compel its formal abandonment.

14. You see, then, to *recapitulate* briefly the chief part of what has been hitherto said, that I invite you to *conceive* the RELATIVE POSITION of any sought point B of space, when compared with any given point A, as being (in what appears to me to be a very easily intelligible and simply symbolizable sense) the GEOMETRICAL DIFFERENCE OF THE ABSOLUTE POSITIONS of those two mathematical points: and that I propose to *denote* it, in this system of symbolical geometry, by writing "*the symbol of the sought point*, MINUS *the symbol of the given point.*" Such is, in my view, the ANALYTIC ASPECT of the compound symbol

$$B - A,$$

if the component symbols A and B be still understood to denote *points:* such is the *primary signification* which I attach *in geometry* to the interposed mark −, when it is regarded as being what I have already called, in general terms, a CHARACTERISTIC OF ORDINAL ANALYSIS.

15. But as you have already also partly seen, the same symbol,

B − A,

may be viewed in a SYNTHETIC ASPECT also. It may be thought of, not only as being the *result of a past analysis*, but also as being the *guide to a future synthesis*. It may be regarded as not merely answering, or as denoting the answer, to the question: *In what Position* is the point B *situated* with respect to the point A? but also this other, which indeed has been already seen to be only the former question *differently viewed: By what Transition* may B be *reached*, if we set out from A?—And to this *other* question *also*, or to this *other view* of the same *fontal Question*, WHERE, I consider the *same symbol*, B − A, to be a fit general representation of the *Answer:* it being reserved for the *context* to decide, whenever a decision may be necessary, *which* of these two related although contrasted views is taken at any one time, in any particular investigation. In its *synthetic aspect*, then, I regard the symbol B − A as denoting "the STEP to B from A:" namely, that step by *making* which, *from* the given point A, we should reach or *arrive at* the sought point B; and *so* determine, generate, mark, or CONSTRUCT that point. This *step* (which we shall always suppose to be a *straight line*) may also, in my opinion, be properly called a VECTOR; or more fully, it may be called "*the vector of the point* B, *from the point* A:" because it may be considered as having for its office, function, work, task, or business, to *transport* or CARRY (in Latin, *vehere*) a *moveable point*, from the given or initial position A, to the sought or final position B. Taking *this view*, then, of the symbol B − A, or adopting now this *synthetic interpretation* of it, and of the corresponding form of words, we may say, generally, for any such conceived rectilinear transport of a moveable point in space, that

"*Step* equals End of Step, *minus* Beginning of Step;"

or may write:

"VECTOR = (End of Vector) − (Beginning of Vector)."

16. Thus, in astronomy, whereas, by the mode of *analytic interpretation* already explained, the phrase,

"Sun's Position *minus* Earth's Position,"

has been regarded (in § 10) as equivalent to the more usual form of words, "Sun's Geocentric Position" (including geocentric distance); we shall *now* be led, by the connected mode of *synthetic interpretation* just mentioned, to regard the same spoken phrase, or the written expression, ☉ – ♁ (where the two astronomical marks, ☉ and ♁, are still supposed to be used to denote the *situations* alone of the two bodies which they indicate), as being equivalent, *in this other view* of it, to what may be called the

"SUN'S GEOCENTRIC VECTOR:"

which DIFFERS from what is called in astronomy the

"*Geocentric Radius-Vector* of the Sun,"

by its INCLUDING DIRECTION, *as well as length*, as an element in its complete signification. In like manner, that equally long but *opposite line*, which may be called, in the same new mode of speaking, the "Earth's Heliocentric *Vector*," may be denoted by the opposite symbol, ♁ – ☉, or expressed by the phrase, "Earth's Position, *minus* Sun's Position;" the Heliocentric *Vector* of Venus will be, on the same plan, symbolically equal or equivalent to the Position of Venus *minus* the Position of the Sun: and similarly in other cases.

17. To illustrate more fully the distinction which was just now briefly mentioned, between the meanings of the "Vector" and the "Radius Vector" of a point, we may remark that the RADIUS-VECTOR, in astronomy, and indeed in geometry also, is usually understood *to have only length;* and therefore to be *adequately expressed by a* SINGLE NUMBER, denoting the *magnitude* (*or length*) *of the straight line* which is referred to by this usual name (radius-vector), as compared with the magnitude of some standard line, which has been assumed as the unit of length. Thus, in astronomy, the Geocentric Radius-Vector of the Sun is, in its mean value, nearly equal to ninety-five millions of miles: if, then, a million of miles be assumed as the standard or unit of length, the sun's geocentric radius-vector is equal (nearly) to, or is (approximately) expressible by, the *number ninety-five:* in such a manner that this *single number*, 95, with the *unit* here supposed, is (at certain seasons of the year) a *full, complete,* and

adequate representation or expression for that known radius-vector of the sun. For it is usually the *sun itself* (or more fully the position of the Sun's centre), and NOT *the Sun's radius-vector*, which is regarded as possessing *also* certain *other* (*polar*) co-ordinates *of its own*, namely, in general, some two angles, such as those which are called the Sun's geocentric right-ascension and declination; and which are merely *associated with* the radius-vector, but *not inherent therein*, nor *belonging thereto*; just as the radius-vector is *itself*, in turn, *associated with* the right ascension and declination, but *not included in them*. Those two angular co-ordinates (or some data equivalent to them) are indeed required to assist in the complete determination of the geocentric position of the SUN ITSELF: but they are *not* usually considered as being in any manner necessary for the most complete determination, or perfect numerical expression, of the Sun's RADIUS-VECTOR. But in the new mode of speaking which it is here proposed to introduce, and which is *guarded from confusion* with the older mode by the *omission of the word* "RADIUS," the VECTOR of the sun HAS (*itself*) DIRECTION, *as well as length*. It is, therefore, NOT *sufficiently characterized by* ANY SINGLE NUMBER, such as 95 (were this even otherwise rigorous); but REQUIRES, *for its* COMPLETE NUMERICAL EXPRESSION, *a* SYSTEM OF THREE NUMBERS; such as the usual and well-known rectangular or polar co-ordinates of the Sun or other body or point whose place is to be examined: AMONG *which* ONE MAY *be* what is called the *radius*-vector; but *if so*, THAT RADIUS MUST (in general) *be associated with* TWO OTHER polar co-ordinates, or determining numbers of some kind, before the VECTOR can be *numerically* expressed. A VECTOR is thus (as you will afterwards more clearly see) a sort of NATURAL TRIPLET (suggested by Geometry): and accordingly we shall find that QUATERNIONS offer an easy mode of *symbolically representing every vector by a* TRINOMIAL FORM ($ix+jy+kz$); which form brings the conception and expression of such a *vector* into the closest possible connexion with Cartesian and rectangular *co-ordinates*.

18. Denoting, however, for the present, a *vector* of this sort, or a rectilinear *step* in space from one point A to another point B, not *yet* by any such trinomial or triplet form, but simply (for

conciseness) by a single and small Roman letter, such as a; and proceeding to *compare*, or equate, these two *equivalent expressions*, or *equisignificant symbols*, a and B − A; we are conducted to the EQUATION,

$$B - A = a;$$

which is thus to be regarded as *here* implying merely that we have *chosen to denote*, concisely, by the simple symbol, or single letter, a, the SAME STEP, or *vector*, which has also been *otherwise denoted*, less briefly, but in some respects more fully and expressively, by the complex symbol B − A. Such is, at least, the *synthetic aspect* under which this *equation* here presents itself; but we may conceive it to occur also, at another time and in another connexion, under an *analytic aspect;* namely, as signifying that the simple symbol a was used to denote concisely the *same ordinal relation* of position, which had been more fully denoted by the complex symbol B − A. Or we may imagine the equation offering itself under a *mixed* (analytic and synthetic) aspect; and as then expressing the *perfect correspondence* which may be supposed to exist between that *relative position* of the point B with respect to the point A, which was originally indicated by B − A, and that *rectilinear transition*, or step, from A to B, which we lately supposed to be denoted by a. Between these different modes of interpretation, the *context* would always be found sufficient to decide, whenever a decision became necessary. But I think that we shall find it more convenient, simple, and clear, during the *remainder* of the present Lecture, to *adhere to the synthetic view* of the equation B − A = a; that is, to regard it as signifying that *both its members*, B − A and a, are *symbols for one common step*, or vector. And generally I propose to employ, *henceforth*, the small Roman or Greek letters, a, b, a′, &c., or α, β, α', &c., with or without accents, as *symbols of steps*, or of *vectors*.

19. But at this stage it is convenient to introduce the employment of another simple *notation*, which shall more distinctly and expressly recognise and mark that *synthetic character* which we have thus attributed to a, considered as denoting the *step* from A to B; in virtue of which synthetic character we have regarded the latter point B as *constructed*, generated, determined, or brought into view, by applying to, or performing on, the former

point A, that ACT OF VECTION or of transport, in which the agent or *operator* is the VECTOR denoted by a. We require a SIGN OF VECTION : a *characteristic of the operation of ordinal synthesis*, by which we have conceived a *sought position* B in space to be *constructed*, as depending on a *given position* A, with the help of a *given vector*, or *ordinal operator*, a, of the kind considered above. And such a CHARACTERISTIC OF ORDINAL SYNTHESIS, or *sign of vection*, is, on that general plan which was briefly stated to you early to-day (in art. 5), supplied by the *mark* +, or by the *word* PLUS, when used in that new sense which has already been referred to in this Lecture, and which may be regarded as *suggested by Algebra*, though it cannot (strictly speaking) be said to be *borrowed from Algebra*, at least as ALGEBRA is commonly viewed. For we shall thus be led to write, as another and an *equivalent form* of the recent equation B − A = a, this other equation, in which *Plus* is introduced, and which is, *in ordinary Algebra also*, a transformation of the equation lately written :

$$B = a + A;$$

while yet, in conformity with what has been already said, we shall now regard it as being the *primary signification* of this last equation, or formula, that "the *position* denoted by B may be REACHED (and, in *that* sense, CONSTRUCTED), by making the TRANSITION denoted by a, *from* the position denoted by A."

20. We shall thus be led to say or to write *generally*, with this (which is here regarded as being the) *primary signification* of PLUS in Geometry, that for any vector or rectilinear step in space,

"Step + Beginning of Step = End of Step;"

or, "Vector + Beginning of Vector = End of Vector:"

the mark + being in fact here regarded, by what has been already said, as being *primarily the sign of vection*, or the characteristic of the *application of a step*, or of a vector, *to a given point* considered as the *Beginning* (of the step, or vector), so as to generate or determine *another point* considered as the *End*. In relation to astronomy, this phraseology will allow us to say that

"Sun's Position = Sun's Geocentric Vector + Earth's position ;"

and the assertion is to be thus interpreted : that if a straight line, agreeing in length and in direction with the line or step in space which we have called in this Lecture the *Sun's Geocentric Vector*, were *applied to the position* occupied by the Earth, so as to *begin there*, this line would *terminate at the Sun*. In exactly the same way, we may say that the "Position of Venus in space" is symbolically expressible as the "Heliocentric Vector of Venus, Plus the Position of the Sun in Space ;" or as the "Geocentric Vector of Venus, *plus* the Position of the Earth ;" and similarly in other cases.

21. All this, as you perceive, is very simple and intelligible; nor can it ever lead you into any difficulty or obscurity, if you will only consent to *use* from the outset, and will take pains to *remember* that you use, the *signs* in the way which I propose; although that way may not be, or rather is certainly not, altogether the same with that to which you are accustomed. Yet you see that it is *not in contradiction* to any received and established use of symbols in Geometry, precisely because *no meaning* is usually attached to any expression of the form, "Line plus point." (Compare 13). Such an expression would be simply *unmeaning*, according to common usage; in short, it would be *nonsense:* but I ask you to allow me to *make it sense*, by giving to it an INTERPRETATION ; which must indeed remain so far a DEFINITION, as that you *may refuse* to accompany me in assigning to the expression in question the signification here proposed. Yet you see that I have sought at least to present that definition, or that interpretation, as *divested of a purely arbitrary character ;* by shewing that it may be regarded as the mental and symbolic *counterpart of another* definitional interpretation, which has already been assigned in this Lecture for another form of spoken and written expression; namely, for the form, "Point minus Point :" which would, according to common usage, be exactly *as* unmeaning, not more so, and not less, than the other. If you yield to the reasons, or motives of analogy, which have been already stated, or suggested, for treating the DIFFERENCE of two Points as a *Line*, it cannot afterwards appear surprising that you should be called upon to treat the SUM of a Line and Point, as being *another Point*.

22. Most fully do I grant, or rather assert and avow, that the

primary signification which I thus propose for + in Geometry, is altogether distinct from that of denoting the operation of *combining two partial magnitudes*, in such a manner as to make up *one total magnitude*. But surely every student of the elements of Algebra is perfectly *familiar with another use of plus*, which is *not less distinct* from such merely *quantitative aggregation*, or simple *arithmetical addition*. When it is granted, as you all know it to be, that "(Negative Seven) + (Positive Three) = (Negative Four)," where the mark + is still *read* as "Plus;" and when this operation of combination is commonly called, as you all know that it *is* called, "Algebraical Addition," and is said to produce an "algebraic sum," although the resulting *number Four* (if we abstract from the adjectives "positive" and "negative") is the *arithmetical difference*, and *not* the arithmetical *sum*, of the *numbers* Seven and Three: there is surely a sufficient *departure, thus authorized already by received scientific usage*, from the merely *popular meanings* of the words "addition," "sum," and "plus," to justify me, or to plead at least my excuse, if I venture on another but scarcely a greater variation from the same first or popular meanings of those words, as indicating (in common language) increase of magnitude; and if I thus *connect them, from the outset* of this new symbolical geometry, *with* CHANGE OF POSITION *in space*.

23. It seems to me then that it ought not to appear a strange or unpardonable *extension* of a phraseology which has *already* been found to require to be extended, in passing from arithmetic to algebra, if I now venture to propose the name of SYMBOLICAL ADDITION for that operation in Geometry, which you have seen that I denote in writing by the sign +; and if I thus speak, for example, in the recent case, of the *Symbolical Addition* of a to A, which operation has been seen to correspond to the *composition*, or *putting together*, in thought and in expression, and therefore to the (conceived or spoken or written) SYNTHESIS, OF THE TWO CONCEPTIONS, *of a* STEP (a) *and the* BEGINNING (A) *of that step:* and NOT (*primarily*) to any synthesis or *aggregation of magnitudes*. Thus if we now agree to give to the *beginning* of the step, or to the *initial position*, the *name* VEHEND (*punctum vehendum*, the point *about to be carried*), because this is the point

on which we *propose to perform the* ACT OF VECTION; and if in like manner the point which is the *end* of the step, or the *final position* (the *punctum vectum*, the point which in this view is regarded as *having been carried*), be shortly called the VECTUM; while the step itself has been already named the VECTOR: we may then establish a technical and *general formula for such symbolical addition in geometry*, which will serve to characterize and express its nature, by saying that, in general,

"VECTUM = VECTOR + VEHEND;"

while the corresponding *general formula for symbolical subtraction in geometry*, with the same new names, will be the following:

"VECTOR = VECTUM − VEHEND."

Nor shall I shrink from avowing my own belief that this general formula, Vectum = Vector + Vehend, may be considered as a TYPE, representing that *primary synthesis in Geometry*, which, earlier and more than any other, ought to be regarded as ANALOGOUS TO ADDITION, in that science, and deserves to be denoted accordingly: namely, the mental and symbolical *addition* (or application) *of a vector to a vehend*, not at all as parts of one magnitude, but as ELEMENTS IN ONE CONSTRUCTION, in order to *generate* as their (mental and symbolical) *sum*, or as the RESULT OF THIS VECTION, or transport, a NEW POSITION IN SPACE, which may be thought of as a *punctum vectum*, or *carried point;* this VECTUM being simply (as has been seen) the *end* of that line, or VECTOR, or *carrying path*, of which the VEHEND is the *beginning*.

24. These relations of *end* and *beginning* may, of course, be *interchanged*, while the straight line AB retains not only its *length*, but even its *situation* in space, although its *direction* will thus come to be *reversed:* for we may conceive ourselves as *returning* from B to A, after having *gone* from A to B. This *path of return*, this backward step, or reversed journey, considered as having for its office to CARRY BACK (*revehere*) a moveable point from B to A, after that point has been first *carried* by the former VECTOR from A to B, may naturally be called, by analogy and contrast, a REVECTOR; and then we shall have this general *formula of revection*,

REVECTOR + VECTUM = VEHEND;

together with this other connected formula:

VEHEND − VECTUM = REVECTOR.

The *symbol* for this *revector* will thus be A − B, if the *vector* be still denoted by the symbol B − A; that is to say, these two *opposite symbols*,

B − A and A − B,

which, in their *analytic aspect*, were formerly regarded by us (see 9) as symbols of two *opposite ordinal relations* in space, *corresponding* to two opposite steps, are *now*, in their *synthetic aspect*, considered as denoting *those two opposite steps themselves;* namely, the Vector and Revector. With reference to the ACT OF REVECTION, the point B, which was formerly called the *vectum*, might now be called the REVEHEND; and then the point A, which was the *vehend* before, would naturally come to receive the name REVECTUM. But I am not anxious that you should take any pains to impress these *last* names on your memory; though I think that it may have been an assistance, rather than a distraction, to have thus briefly suggested them in passing.

25. If in the general formula lately assigned (in 23) for symbolical *addition* in geometry, namely the formula, vector + vehend = vectum, we *substitute* for *vector* its *value*, or equivalent expression, namely, vectum − vehend, as given by the corresponding general formula already assigned (in same art. 23) for symbolical *subtraction;* we shall thereby *eliminate* (or get rid of) the *word* "vector," in the sense that this word will *no longer appear* in the *result* of this subtraction; which result will be the equation,

Vectum − Vehend + Vehend = Vectum.

In symbols, the corresponding elimination of the letter a, between the two equations,

B − A = a,   a + A = B,   (18, 19)

gives, in like manner, the result: B − A + A = B. In ordinary Algebra, not only does the same result hold good, but it is said to be *identically true*, and the equation which expresses it is called an IDENTITY; and in the present Symbolical Geometry it may *still* be called by that name: in the sense that *its truth does not depend*, in any degree, *on the positions of the two points*, A, B;

but only on the *general connexion*, or contrast, *between the two* OPERATIONS *of ordinal* ANALYSIS and SYNTHESIS, which are here marked by the signs − and +. For the formula $B - A + A = B$, or more fully, $(B - A) + A = B$, may be considered as expressing, in the present system of symbols, that if the position A be *operated on* (synthetically) by what has been called the symbolical *addition* (or application) of a suitable *vector*, namely $B - A$, it will be *changed* to the position B; *such* SUITABLE OPERATOR $(B - A)$ being precisely *that vector* which is conceived to have been *previously discovered* (analytically) by what we have called the symbolical *subtraction* of the proposed *vehend* A from the *vectum* B. Until the points A and B are in some degree known, or particularized, the line $B - A$ must also be unknown, or undetermined: yet must this line be such (in virtue of its definition, or of the rule for its construction) as to conduct, or to be capable of conducting, *from* the point A *to* the point B. We *know this*, and this is *all* we know, about that line, in general: and we *express* it by the general equation or identity, $B - A + A = B$.

26. In like manner, if we eliminate the word "Vectum," or the letter B, between those general equations or formulæ of symbolical addition and subtraction in geometry which have been already assigned, we arrive at this *other identity*,

$$\text{Vector} + \text{Vehend} - \text{Vehend} = \text{Vector};$$

or in symbols,

$$a + A - A = a; \text{ or more fully, } (a + A) - A = a:$$

which must hold good for *any* vehend A, and *any* vector a. The same result would evidently be true, and identical, in ordinary Algebra also: but it is *here* to be *interpreted* as signifying that if, from *any point* A, we make *any rectilinear step* a, and then *compare the end* $a + A$ of this rectilinear step *with the beginning* A, we shall be *reconducted*, by this *analysis* of the relative position of these two points, to the consideration and determination of the *same straight line* a, which is supposed to have been *already* employed in the previous construction, or *synthesis*. You will find hereafter that *many other* instances occur, on which, however, it will be impossible in these Lectures long to delay, or perhaps often even to notice them at all, where equations or results, that are true in ordinary Algebra, hold good *also* in this new sort of Symbolical Geometry; although generally regarded in *new lights*, and bearing new (if not enlarged) *significations*.

27. In all that has yet been said respecting the *acts* of "vection" and "revection," or the *lines* "vector" and "revector," we have *hitherto* had occasion to consider *only two points*; namely, those which have been above named the "vehend" (or the revectum) A, and the "vectum" (or revehend) B. Let us *now* introduce the consideration of a *third point*, C, which we shall *not generally* suppose to be situated *on* the straight line AB, nor on that line either way *prolonged*; *but* rather so that the three points ABC may admit (for the sake of greater generality) of being regarded as the three corners of a *triangle*. And let us conceive that the former act of *vection*, whereby a moveable point was before imagined to have been carried from the position A to the position B, is now *followed* by *another* act of the same kind, that is to say, by an immediately *successive vection*, which we shall call on that account (from the Latin word *provehere*) a PROVECTION: whereby the *same moveable point* is now CARRIED FARTHER, though *not* (generally) in the *same straight line*, but along a *new* and *different straight line*; and is in this manner transported from the position B to the position C. We shall thus be led to consider the line C − B as being a new and *successive vector*, which may conveniently be called, on that account, a PROVECTOR: the point B, which had been named the *Vectum*, may now be *also* named the PROVEHEND, with reference to the new *act of provection* here considered, and which *begins* where the old act of vection *ends*: while, with reference to the same new act of transport, or provection, the point C will naturally come to be called (on the same plan) the PROVECTUM. And thus we shall have, for any such successive vection, the formula,

$$\text{Provector} + \text{Vectum} = \text{Provectum};$$

as also the connected formula,

$$\text{Provector} = \text{Provectum} - \text{Vectum}.$$

It is worth noticing here, that if we *substitute*, in the first of these two new equations, for the word "Vectum," its *value*, or equivalent expression, namely, "Vector + Vehend" (23), we shall be thereby led to write this other *formula of provection:*

Provector + Vector + Vehend = Provectum.

28. In symbols, if we write the equation

$$\text{C} - \text{B} = \text{b},$$

so that the small Roman letter b shall here be used as a short symbol for the provector, while a remains, as before, a symbol for the vector, and satisfies still the equation (18),

$$\text{B} - \text{A} = \text{a};$$

we shall then have not only, as before (19),

$$\text{B} = \text{a} + \text{A},$$

but also, in like manner,

$$\text{C} = \text{b} + \text{B}.$$

And then, by *eliminating* B, we shall have also this other formula,

$$\text{C} = \text{b} + \text{a} + \text{A};$$

or more fully,

$$\text{C} = \text{b} + (\text{a} + \text{A}).$$

We may also write, without introducing the symbols a and b,

$$\text{C} = (\text{C} - \text{B}) + \{(\text{B} - \text{A}) + \text{A}\};$$

because the second member of this equation may be reduced (by 25) to (C − B) + B, and therefore to C; or, more concisely, we may write,

$$\text{C} = (\text{C} - \text{B}) + (\text{B} - \text{A}) + \text{A};$$

which gives again, in *words*,

Provectum = Provector + Vector + Vehend.

The last symbolic formula (with A, B, C) is in common Algebra an *identity;* and we see that is here also at least a *general equation* (*of provection*), which holds good for *any three points of space*, A, B, C, *independently of the positions* of those points, and in virtue merely of the *laws* of composition and interpretation of the *symbols*, or in virtue of the *relations* between the (conceived) *operations* which the signs denote: so that it may perhaps be called here (compare 25) a GEOMETRICAL IDENTITY.

29. Astronomically, we may conceive C to denote the position of the centre of a planet; while A and B denote still the positions

of the centres of the earth and sun: and then, while the *vector* (B – A) is still the geocentric vector of the sun, the *provector* (C – B) will be the heliocentric vector of the planet. And in a phraseology already explained, we shall not only have as before (20) the equation,

Sun's position = Sun's geocentric vector + Earth's position,

and in like manner,

Planet's position = Planet's heliocentric vector + Sun's position,

but also, by a *combination* of these two assertions, or phrases, or equations, which combination is effected by *substituting* in the latter of them the *equivalent* for the "Sun's position" which is supplied by the former, we shall be able to conclude the correctness of the following *other* assertion (in this general system of expressions):

"Planet's position = Planet's Heliocentric Vector
+ Sun's Geocentric Vector + Earth's Position."

30. Instead of thus imagining a moveable point to be *carried in succession*, first along *one* straight line (B – A) from A to B, and then along *another* straight line (C – B) from B to C, which lines have been supposed to be in general *two successive sides*, AB, BC, of a triangle ABC; we may conceive the moveable point to be CARRIED ACROSS, by the straight line (C – A) or *along the third side*, or *base*, AC, of the same triangle, from the original position A to the final position C. And this new act of transport may be called a TRANSVECTION (from the Latin word *transvehere*, to carry across); while the line C – A, when viewed as such a *cross-carrier*, may be called a TRANSVECTOR: and the points A and C, which were before termed the Vehend and the Provectum, will now come to be called, with reference to this *new act* of transport, or *transvection*, the TRANSVEHEND and the TRANSVECTUM, respectively. *Comparing* then the *names* of the three points, we shall have the following new *equations*, or *expressions of equivalence* between them:

Transvehend = Vehend = A;
Provehend = Vectum = B;
Transvectum = Provectum = C:

each corner of the triangle ABC being thus regarded in two different *views*, or presenting itself in two different *connexions*, and receiving *two names* in consequence thereof, on account of its relations to some *two* out of the three different *acts*, or operations, of vection, provection, and transvection. And by a suitable selection among these names for A and C, the following equation (see 25),

$$\mathrm{C} = (\mathrm{C} - \mathrm{A}) + \mathrm{A},$$

may now be *translated* as follows:

Provectum = Transvector + Vehend.

31. Combining this result with another recent expression for the Provectum (at end of 27), we see that we may now enunciate the equation:

Provector + Vector + Vehend = Transvector + Vehend ;

each member of this last equation being an expression for one and the *same point*, namely the Provectum, or the point C. And when this equation had once been enunciated, under the form just now stated, an *instinct of language*, which leads to the avoidance of repetition in ordinary expression, and so to the abridgment of discourse, when such abridgment can be attained without loss of clearness or of force, might of itself be sufficient to *suggest* to us the *suppression* of the words "plus vehend," which occur at the end of *each* member of the equation (+ being always *read* as *plus*). In this way, then, we may be led to enunciate the following *shorter* formula:

"PROVECTOR + VECTOR = TRANSVECTOR ;"

this latter formula (which we shall find to be a very important one) being thus considered, *here*, as nothing more than an ABBREVIATION of that longer equation, from which it is supposed to have been in this way derived.

32. In symbols, if we write

$$\mathrm{C} - \mathrm{A} = \mathrm{c}$$

thus making c a symbol of the transvector; and if we compare the expression hence resulting for C, namely (see 19),

$$\mathrm{C} = \mathrm{c} + \mathrm{A},$$

with the expression already found (in 28),

$$\mathrm{C} = \mathrm{b} + \mathrm{a} + \mathrm{A};$$

we shall thus be led to the equation,

$$b + a + A = c + A,$$

which we may (in like manner) be tempted to *abridge*, by the *omission* of + A at the end of *each* of its two members ; and so to reduce it to the shorter form,

$$b + a = c,$$

which agrees with the recent result, Provector + Vector = Transvector (31) ; because a, b, c denote here the vector, provector, and transvector, respectively. Or, without introducing these symbols a, b, c, if we compare a recent expression for C, namely (see 28),

$$C = (C - B) + (B - A) + A,$$

with this other expression (compare 25),

$$C = (C - A) + A,$$

and *suppress* + A *in both*, as before, we shall thus be conducted to the *general equation*, or *geometrical* (as well as algebraical) IDENTITY :

$$(C - B) + (B - A) = (C - A) ;$$

which again agrees with the result (of 31),

"Provector + Vector = Transvector."

33. In a phraseology suggested by astronomy, and partly employed already in this Lecture, we have on the one hand (as in 29),

Planet's Position = Planet's Heliocentric Vector
+ Sun's Geocentric Vector + Earth's Position ;

and on the other hand (see 20),

Planet's Position = Planet's Geocentric Vector + Earth's Position.

Comparing these two different expressions for the position of the planet in space, and suppressing a part which is common to both, namely, the words

"Plus Earth's Position,"

we shall be led to say that

"Planet's Heliocentric Vector
+ Sun's Geocentric Vector
= Planet's Geocentric Vector :"

where the geocentric vector of the planet is to be regarded as the *transvector* in the triangle, if the planet's heliocentric vector be the *provector*, while the geocentric vector of the sun is the original *vector* itself.

34. Since (by 27),

Provector = Provectum − Vectum,

while (by 30 and 23),

Provectum = Transvector + Vehend,

and

Vectum = Vector + Vehend,

we have the equation

Provector = (Transvector + Vehend)
− (Vector + Vehend);

which may conveniently be *abridged* to the following formula:

"PROVECTOR = TRANSVECTOR − VECTOR."

Thus, in astronomy, we may say that

"Planet's Heliocentric Vector
= Planet's Geocentric Vector
− Sun's Geocentric Vector;"

regarding the second member of this equation as an *abridgment* for the following expression:

(Planet's Geocentric Vector + Earth's Position)
− (Sun's Geocentric Vector + Earth's Position);

which we know to be equivalent, in the phraseology of the present Lecture, to

"Planet's Position − Sun's Position;"

and therefore to "Planet's Heliocentric Vector," as above.

35. In symbols, because (by 28, 32, 19),

$$\mathrm{b} = \mathrm{C} - \mathrm{B}, \quad \mathrm{C} = \mathrm{c} + \mathrm{A}, \quad \mathrm{B} = \mathrm{a} + \mathrm{A},$$

we have the equation

$$\mathrm{b} = (\mathrm{c} + \mathrm{A}) - (\mathrm{a} + \mathrm{A});$$

which may be *abridged* to the following:

$$\mathrm{b} = \mathrm{c} - \mathrm{a}.$$

This signification of $\mathrm{c} - \mathrm{a}$ allows us also to extend to geometry the algebraical *identity:*

$$(\mathrm{C} - \mathrm{A}) - (\mathrm{B} - \mathrm{A}) = (\mathrm{C} - \mathrm{B});$$

and generally it will be found to prepare for the establishment of a complete *agreement* between the *rules* of ordinary Algebra and those of the present Symbolical Geometry, so far as *addition* and *subtraction* are concerned. Thus, if we compare the two equations (32, 35),

$$\mathrm{c} = \mathrm{b} + \mathrm{a}, \quad \mathrm{b} = \mathrm{c} - \mathrm{a},$$

we find that generally, for any two *co-initial vectors*, a, c, we may write (as in ordinary Algebra),

$$(\mathrm{c} - \mathrm{a}) + \mathrm{a} = \mathrm{c};$$

and that for any two *successive vectors*, a, b, we have also (as in Algebra):

$$(\mathrm{b} + \mathrm{a}) - \mathrm{a} = \mathrm{b};$$

which new *geometrical identities* are of the *same forms* as some others that were lately considered (in 25, 26), namely,

$$(\mathrm{B} - \mathrm{A}) + \mathrm{A} = \mathrm{B}; \quad (\mathrm{a} + \mathrm{A}) - \mathrm{A} = \mathrm{a}.$$

Indeed they have with these a very *close connexion*, as regards their *significations* too, arising out of the way in which they have been above obtained; yet because A, B, C have been used as symbols of *points*, but a, b, c as symbols of *lines*, it would have been illogical and hazardous to have *confounded* these two pairs of equations, or identities, with each other; or to have regarded the truth of the one pair as an *immediate* consequence of the truth of the other pair.

36. We see, however, that the original VIEW which has been proposed, in the present Lecture, for the PRIMARY SIGNIFICATIONS of + and − in geometry, as entering *first* into expressions of the (unusual) forms "*Line plus Point*" and "*Point minus Point*," conducts, simply enough, when followed out, to interpretations of expressions of the (more common) forms "*Line plus Line*," and "*Line minus Line*:" and that thus, from what we have regarded as the PRIMARY ACTS *of synthesis and analysis* (of points) *in geometry*, arise a SECONDARY SYNTHESIS and a SECONDARY ANALYSIS (of lines), which correspond to the *composition and decomposition of vections* (or of motions); and which are symbolized by the two general formulæ already assigned (in 31, 34), namely,

$$\text{Transvector} = \text{Provector} + \text{Vector},$$

and

$$\text{Provector} = \text{Transvector} - \text{Vector}.$$

The first formula asserts that of any two *successive vectors*, or directed lines (the second or *added* line being conceived to *begin* where the first line *ends*), the GEOMETRICAL SUM is the line drawn from the beginning of the first to the end of the second line. The second formula asserts, that of any two *co-initial vectors* (or directed lines), the GEOMETRICAL DIFFERENCE is the line drawn from the end of the *subtrahend* line to the end of the line from which it is subtracted. The *sum* and the *difference* of two directed *lines* are thus *two other lines* having direction ; and the *geometrical rules* for determining them are found to *coincide in* THIS *theory, as in several* OTHERS ALSO, with the rules of COMPOSITION *and* DECOMPOSITION *of* MOTIONS (*or of forces*). For, although it would be unsuited to the plan and limits of these Lectures to enter deeply, or almost at all, into the *history* of those speculations to which their subject is allied, yet it seems proper to acknowledge distinctly here, as I am very happy to do, that (whatever may be thought of the foregoing *general views* respecting + and –), the recognition of an ANALOGY between ADDITION *and* SUBTRACTION *of directed* LINES, on the one hand, and *composition and decomposition of* MOTIONS on the other hand, is *nothing private or peculiar to myself.* Indeed, the existence of this fundamentally important *analogy* has, in different ways, presented itself to SEVERAL OTHER thinkers, starting from various points of view, in many parts of the world, during the present century : so much so, that it may by this time be well nigh considered to have acquired, in the philosophy of geometrical science, what I cannot doubt its possessing still more fully in time to come, the character of an admitted and established truth, a fixed and settled principle. But of those more novel and hitherto less participated views, respecting the MULTIPLICATION and DIVISION of such *directed lines* in geometry, on which the theory of QUATERNIONS is founded, I perceive that our time requires that we should postpone the consideration to the next Lecture of this Course : for which, however, I indulge myself meanwhile in hoping, that what has been laid before you to-day will be found to have been an useful, and indeed a necessary preparation.

LECTURE II.

37. You have had laid before you, Gentlemen, in the foregoing Lecture, a statement or at least a sketch of those *general views*, respecting the *primary significations* of the marks

$$+ \text{ and } -,$$

or of the words PLUS and MINUS, with which views, in the Calculus of Quaternions, I connect the two corresponding *operations* of Addition and Subtraction in Geometry. With me, as you have seen, the primary geometrical operation which has been denoted by the usual mark −, and the one for which I have ventured to employ the familiar *name* SUBTRACTION, though guarded sometimes by the epithet *symbolical*, consists in a certain *ordinal Analysis* of the *position* of a mathematical point in space. This Analysis is *performed*, as you have seen, through the *comparison* of the position of the point proposed for inquiry, with the position of *another* mathematical point; and it is *pictured*, or represented, by the TRACTION (or drawing) of a straight LINE, from the given to the sought position; from the *analyzer* point A, to the *analyzand* point B: from the one which is regarded as being comparatively simple, familiar, or given, to the other which is (for the purposes of the inquiry) accounted to be comparatively complex, unknown, or sought. In this way, the symbol B − A has come with us to denote *the straight line from* A *to* B; the point A being (at first) considered as a *known* thing, or a *datum* in some geometrical investigation, and the point B being (by contrast) regarded as a *sought* thing, or a *quæsitum:* while B − A is at first supposed to be a representation of the *ordinal relation in space*, of the sought point B to the given point A; or of the geometrical DIFFERENCE of those *two points*, that is to say, the difference of their two POSITIONS in space; and this *difference* is supposed to be exhibited or constructed by a straight *line*. Thus, in the astronomical example of earth and sun, the line B – A has been seen to extend *from the place of observation* A (the earth), *to the place of the observed body* B (the sun); and to serve to CONNECT, at least in thought, the latter position with the former.

38. Again you have seen that with me the *primary* geometrical operation denoted by the mark +, and called by the name ADDITION, or more fully, symbolical Addition, consists in a certain correspondent *ordinal* SYNTHESIS of the position of a mathematical point in space. Instead of *comparing* such a position, B, with another position A, we *now* regard ourselves as *deriving* the one position from the other. The point B *had been* before a *punctum analyzandum ;* it *is* now a *punctum constructum.* It was lately the *subject* of an analysis ; it is now the *result* of a synthesis. It was a *mark* to be aimed at ; it is now the *end* of a flight, or of a journey. It was a thing *to be investigated* (analytically) by our studying or examining its position ; it is now a thing which *has been produced* by our operating (synthetically) on another point A, with the aid of a certain *instrument*, namely, the straight line B – A, regarded now as a VECTOR, or carrying path, as is expressed by the employment of the SIGN OF VECTION, +, through the general and identical formula :

$$(\text{B} - \text{A}) + \text{A} = \text{B}.$$

That other point A, instead of being now a *punctum analyzans*, comes to be considered and spoken of as a *punctum vehendum ;* or more briefly, and with phrases of a slightly less foreign form, it was an *analyzer*, but is now a VEHEND ; while the point B, which had been an *analyzand*, has come to be called a VECTUM, according to the general formula :

$$\text{Vector} + \text{Vehend} = \text{Vectum};$$

where *Plus* is (as above remarked) the Sign of Vection, or the *characteristic of ordinal synthesis*. From serving, in the astronomical example, as a *post of observation*, the earth, A, comes to be thought of as the *commencement of a transition*, B – A, which while thus *beginning* at the earth is conceived to *terminate* at the sun ; and conversely the sun, B, is thought of as occupying a situation in space, which is not now proposed *to be studied* by observation, but is rather conceived as one which *has been reached*, or arrived at, by a journey, transition, or transport of some moveable point or body *from* the earth, *along* the geocentric vector of the sun. I think that this brief review, or *recapitulation*, of some of the chief features or main elements of the view already taken, of the *operations* of Addition and Subtraction, or of the marks + and −, will be found to have been not useless, as preparatory to our entering now on the consideration of the *analogous view* which I take of the operations of Multiplication and Division, or of the marks × and ÷ in Geometry.

39. The Analysis and Synthesis, hitherto considered by us, have been of an ORDINAL kind; but we now proceed to the consideration of a different and a more complex sort of analysis and synthesis, which may, by contrast and analogy, be called CARDINAL. As we before (analytically) *compared a* POINT, B, *with a point* A, with a view to discover the *ordinal relation in space* of the one point to the other; so we shall now go on to *compare* one *directed line*, or *vector*, or RAY, β, *with another ray*, a, to discover what (in virtue of the contrast and analogy just now referred to) I shall venture to call the *cardinal relation of the one ray to the other*, namely, (as will soon be more clearly seen), *a certain complex relation of length and of direction.* As one among the reasons for the adoption of such a phraseology which may admit of being most easily and familiarly stated, while the statement of it will serve, at the same time, as an initial preparation, or introduction, to questions or cases of greater difficulty or complexity, let me remind you that when the condition $\beta = a + a$ is satisfied, it is then permitted, by ordinary usage, to write also $\beta \div a = 2$; the *quotient* of β, *divided* by a, being, in this case, equal to the *cardinal number*, two. Under the same simple condition, it is, as you know, allowed by custom to write also $\beta = 2 \times a$; and to say that the *multiplication* of a, by the same cardinal number, two, *produces* β. Now I think that we may not improperly say that we have here, in the division, *cardinally analyzed* β, as a *cardinal analyzand*, with respect to a, as a *cardinal analyzer*; and that we have *obtained* the cardinal *number*, or *quotient*, 2, as the *result* of this *cardinal analysis*; while, in the converse process of multiplication, we may be said to have

employed the same number, *two*, as a *cardinal operator*, or as the *instrument* of a *cardinal synthesis*, which instrument or operator thus serves as a multiplier, or as a *factor*, to *generate* or to *construct* β, as a *product* or as a *factum*, from a as a *multiplicand* or *faciend*. In so simple an instance as this, it might be better, indeed, to abstain from the use of any part of this phraseology which should seem in any degree unusual; but there appears to me to be a convenience in applying the foregoing modes of expression to the much *more general* case, where it is proposed to *compare* ANY ONE *ray*, β, *with* ANY OTHER *ray*, a, *with a view to discover the complex* RELATION OF LENGTH AND OF DIRECTION of the former to the latter ray; or, conversely, to *construct* or *generate* β *from* a, by making *use* of such a relation.

40. In adopting, then, from ordinary algebra, as we propose to do, the general and identical formula,

$$\beta \div a \times a = \beta,$$

we shall now suppose that $\beta \div a$ denotes *generally* a certain *metrographic relation* of the ray β to the ray a, *including* at once, as its *metric element*, a *ratio of length to length*, and *also*, as its *graphic element*, a *relation of direction to direction*. The *act* or *process of discovering* such a metrographic relation, denoted by the symbol $\beta \div a$, we shall call, generally, the CARDINAL ANALYSIS of β, as an analyzand, by a as an analyzer. And the *converse* act of *employing* such a cardinal relation, when already found or given, so as to form or to *construct* β by a suitable operation on a, namely, by *altering its length in a given ratio*, and by *causing its direction to revolve through a given angle, in a given plane*, and *towards a given hand*, we shall call a CARDINAL SYNTHESIS. The cardinal analysis above mentioned, we shall also call the DIVISION, or, sometimes more fully, the *symbolical* division of the ray β *by* the ray a; and the usual name, QUOTIENT, shall be occasionally applied by us to the *result* of this division, that is, to the metrographic relation denoted above by the symbol $\beta \div a$, and supposed to be *found* by that cardinal analysis, of which the mark $\div$ is thus the *sign*, or the CHARACTERISTIC. In like manner to that converse cardinal synthesis, of which the *characteristic* is here supposed to be the mark $\times$, we shall give (from the analogy which it will be found to possess to the operation commonly so called) the name of MULTIPLICATION, or sometimes, more fully, that of *symbolical* multiplication. And when, after writing an *equation* of the form

$$\beta \div \alpha = q,$$

we proceed to *transform* it into this other equation,

$$q \times \alpha = \beta,$$

(by an application of a general formula lately cited), we shall say that q has been *multiplied into* α, or (sometimes) that α has been multiplied *by* q; *avoiding*, however, to say, conversely, that q has been multiplied *by* α, or α *into* q. Thus q, which had, relatively to the cardinal *analysis* ($\div$), been regarded as a *quotient*, will come to be regarded, and to be spoken of, with reference to the cardinal *synthesis* ($\times$), as a *multiplier*, or as a FACTOR; while β may still be called, as above, a PRODUCT, or a FACTUM: and α may, by contrast, be called a *multiplicand*, or a FACIEND.

41. Without *yet* entering more *minutely* into the consideration of the *precise* force, and *full* geometrical signification, of that *act* or operation which has here been called *Multiplication*, or FACTION; it may be seen already that the *general type* of this process of *cardinal synthesis* is, in the present phraseology, contained in the following technical statement, or *formula*:

$$\text{FACTOR} \times \text{FACIEND} = \text{FACTUM};$$

where we shall still *read*, or translate, the mark $\times$ by the *word* "INTO." It is clear also that the converse process of what has been above called *Division*, or *cardinal analysis*, has, in like manner, *its* general type in the reciprocal formula,

$$\text{FACTUM} \div \text{FACIEND} = \text{FACTOR};$$

where the mark $\div$ may still be translated, or read, as equivalent to the word "BY." And it is evident that these two general and technical assertions, respecting the kind of (symbolical) Multiplication and Division in Geometry which we here consider, are closely analogous to the two corresponding formulæ, already assigned (in art. 23), as types of those earlier operations in geometry which were there called (symbolical) Addition and Subtraction, namely, the two following:

Vector + Vehend = Vectum;

Vectum − Vehend = Vector.

42. It is easy to push this analogy farther with clearness and advantage. We have, for instance, the general formula of identity,

Factum ÷ Faciend × Faciend = Factum;

which corresponds to the identity (of art. 25),

Vectum − Vehend + Vehend = Vectum.

More concisely and symbolically, the written identity (of art. 40), $\beta \div a \times a = \beta$, corresponds exactly to the earlier identical formula (of same art. 25), $\mathrm{B} - \mathrm{A} + \mathrm{A} = \mathrm{B}$. Each is to be considered as telling us *nothing* whatever respecting the *points* or *lines* which *seem* to be compared, and of which the symbols enter into the formulæ; but only as expressing, each in its own way, a *general* relation, of a *metaphysical* rather than of a *mathematical* kind, between the *intellectual* operations, or *mental acts*, of *Synthesis* and of *Analysis*. For each of these technical formulæ may be regarded as an embodiment, in one or other of two different mathematical forms, of the general and abstract principle, that *if the* KNOWLEDGE *previously* ACQUIRED, *by any suitably performed* ANALYSIS, *be afterwards suitably* APPLIED, *by the Synthesis answering to that Analysis, it will conduct to a suitable* RESULT: *which result, thus constructed by this synthesis, will be the very* SUBJECT (whether point, or line, or other thing, or thought) *which had been analyzed before.* Or that whatever has been *found* by Analysis may afterwards be *used* by Synthesis (or at least may be *conceived* to be so used); and that the thing or thought which is *produced* (or *re*-produced) by this *synthetic* process, will be the *same* with that which had been *examined* or submitted to *analysis* previously.

43. Corresponding remarks apply to the written and spoken identities,

$$q \times a \div a = q,$$

and

Factor × Faciend ÷ Faciend = Factor;

which are obviously analogous to the identical formulæ (of 26),

$$\mathrm{a} + \mathrm{A} - \mathrm{A} = \mathrm{a},$$

and

Vector + Vehend − Vehend = Vector.

In fact these technical formulæ may be regarded as being merely so many different mathematical modes of embodying the general and abstract principle, that *whatever specific instrument* (a or q) of any *known sort* of synthesis (+ or ×), is conceived to have been *previously used*, in operating on a *known subject* (A or a), may be conceived to be *afterwards found*, by the converse act of analysis (− or ÷).

44. After comparing any two rays, α and β, with each other by cardinal analysis, *in one order* (β with α), we may choose to compare *again* the same two rays among themselves, but in the *opposite* order (α with β); *exchanging* thus the places of the analyzer and analyzand, in the process of the cardinal analysis. The relations, or the quotients, thus obtained, and denoted by the symbols $\beta \div \alpha$ and $\alpha \div \beta$, may be called *reciprocal cardinal relations*, or *reciprocal quotients;* as (in art. 9) we called B − A and A − B the symbols of two *opposite ordinal relations*. Considered as *reciprocal operators*, or as *inverse factors*, the same two symbols, $\beta \div \alpha$ and $\alpha \div \beta$, may be said to denote, respectively, a *Factor* and its answering REFACTOR; as the two *opposite steps* denoted by B − A and A − B, were called (in art. 24), in respect of each other, by the names of *Vector* and REVECTOR. And in reference to this *act* of REFACTION, we might call β the REFACIEND, and α the REFACTUM; as B has been called (in 24) the REVEHEND, and A has been called the REVECTUM.

45. We shall now proceed to make a further extension of this sort of phraseology; of which extension the deficiency (whatever it may be) in elegance will, it is hoped, be compensated by the systematic convenience which will arise from its resemblance or analogy to the language of the former Lecture; and from the consequent illustration which may be thrown on one set of thoughts by their being brought into contact or juxtaposition with another set, which other has been already considered. I venture, therefore, to propose to you to speak now, or to allow me to speak, of an act of PROFACTION as being performed, when, after having constructed a *second* ray β, *from* a *first* ray α, by a *first act of faction*, or of cardinal synthesis, such as has been already spoken of, we proceed to the construction of a *third* ray, γ, *from* the *second* ray, β, by the performance of a *new* and *successive*

act of synthesis, of the *same general kind* as before; although this *new act of faction*, by which we pass to γ from β, *may not* (and generally will not) be a *simple continuation*, or a *mere repetition*, of the first factor act, but *may* (and generally will) be performed with a quite *different factor* as its instrument. And then that *third act* of the same sort, which is able of itself alone to *replace*, or is *singly equivalent to, the system of these two successive acts* of faction and profaction, may be called an act of TRANSFACTION.

46. Writing then the equation,

$$\gamma \div \beta = r,$$

and, therefore, also (see art. 40),

$$\gamma = r \times \beta,$$

we shall call r the PROFACTOR, because it is the instrument or agent in the second successive act, above mentioned, of cardinal synthesis, or is the *operator* of that *profaction*, by which the ray γ is generated or constructed from the ray β, after β has been already constructed from a by the former act of faction. And with reference to the same *successive* faction, or *pro*-faction, we shall call β the PROFACIEND, and γ the PROFACTUM; in such a manner that we shall be able to enunciate the following *formula of profaction:*

$$\text{Profactor} \times \text{Profaciend} = \text{Profactum};$$

together with the converse formula,

$$\text{Profactum} \div \text{Profaciend} = \text{Profactor};$$

as in the foregoing lecture we might have said in speaking of *provection*,

$$\text{Provector} + \text{Provehend} = \text{Provectum};$$

and

$$\text{Provectum} - \text{Provehend} = \text{Provector}.$$

47. And inasmuch as the same ray, β, is here considered and named as the *Profaciend*, which had before been named, in a different connexion, the *Factum*, we may *substitute* for the word "Profaciend," in the first verbal formula of the last article, the word "Factum," so as to obtain this other formula (analogous to one of art. 27),

Profactor × Factum = Profactum.

We may also proceed to substitute here for "Factum," its *value* (assigned by art. 41), namely, the equivalent expression,

Factor × Faciend ;

and so obtain this *other* general *formula of profaction* (analogous to the formula of provection at the end of art. 27),

Profactor × Factor × Faciend = Profactum.

In symbols, if,

$$\beta = q \times a, \text{ and } \gamma = r \times \beta,$$

we may write, by *elimination* of β,

$$\gamma = r \times q \times a.$$

Or, because $q = \beta \div a$, $r = \gamma \div \beta$, we may write the *identical* formula (analogous to one in art. 28),

$$\gamma = (\gamma \div \beta) \times (\beta \div a) \times a.$$

48. Conceiving, in the next place (see end of art. 45), that the *two successive acts* of faction and profaction are *replaced* by a *single act* of the same sort, *equivalent to the system of these two*; namely, by a certain act of *transfaction*, in which the Operator, or the TRANSFACTOR, shall be (for the present) denoted by the letter s ; we may then write

$$\gamma = s \times a ; \ \gamma \div a = s ;$$

and with respect to this act of *transfaction*, may call a the TRANSFACIEND, and γ the TRANSFACTUM. We shall thus have the two general and reciprocal formulæ,

Transfactor × Transfaciend = Transfactum ;
Transfactum ÷ Transfaciend = Transfactor ;

with two identities, deducible by the comparison of these. And because the ray γ is here at once the *transfactum* and the *profactum*, according as we consider one or the other of the two operations of which that ray is the result ; while the other ray, namely, a, is at once the *faciend* and the *transfaciend* ; we may enunciate this other general formula (compare art. 30),

Transfactor × Faciend = Profactum ;

as, in symbols, we have the identity,

$$(\gamma \div a) \times a = \gamma.$$

49. *Equating* then the two expressions for the Profactum, or for γ, found in the two last articles, we have, in symbols (compare 32), the formula

$$(\gamma \div a) \times a = (\gamma \div \beta) \times (\beta \div a) \times a;$$

and in words (compare 31) we have this general enunciation,

Transfactor × Faciend = Profactor × Factor × Faciend.

Hence (compare again the same articles 31 and 32), we may be naturally led to adopt the two following *abbreviated* forms of assertion, namely, in symbols,

$$(\gamma \div a) = (\gamma \div \beta) \times (\beta \div a);$$

and in words,

TRANSFACTOR = PROFACTOR × FACTOR.

You see, then, that each of these two last equations (of which the first is true and identical in ordinary algebra also) is *here* regarded as an ABRIDGED FORM, which is to be *restored* (where required) to its complete original significance, or full and developed expression, by *restoring the suppressed symbols*, $\times a$, or by restoring the *suppressed words*, "Into Faciend;" exactly as it was supposed (in the articles recently referred to), that the identical equations,

$$(C - A) = (C - B) + (B - A),$$

and

Transvector = Provector + Vector,

were *abridged forms*, which were to be *interpreted*, or restored to *their* full meanings, by *restoring* the symbols + A at the right hand of each member of the one equation, or the words "Plus Vehend" after each member of the other. And we see that, on the present plan, as well as in ordinary algebra, whenever we have (as above supposed)

$$q = \beta \div a; \quad r = \gamma \div \beta; \quad s = \gamma \div a;$$

and when we have, therefore, also the equation (in which each member is $= \gamma$, and the ray a is conceived to have some actual length),

$$s \times a = r \times q \times a;$$

we may then *abbreviate* this last equation to the shorter form,

$$s = r \times q.$$

50. In like manner, because, under the conditions recently mentioned, we have

$$r = \gamma \div \beta = (s \times a) \div (q \times a),$$

or

$$\text{Profactor} = (\text{Transfactor} \times \text{Faciend}) \div (\text{Factor} \times \text{Faciend}),$$

we may also agree to *write*, more concisely (compare art. 35),

$$r = s \div q,$$

and also to *say* (compare art. 34),

PROFACTOR = TRANSFACTOR ÷ FACTOR.

And thus we shall be conducted (as in ordinary algebra) to the following identical formulæ (compare 35),

$$(s \div q) \times q = s; \quad (r \times q) \div q = r;$$

which have, indeed, a very close connexion, both of form and of signification, with the identical equations (of articles 40, 43),

$$(\beta \div a) \times a = \beta; \quad (q \times a) \div a = q;$$

yet which are *not*, in the present system, to be *confounded* therewith. For a, β, γ, have been supposed to be *rays*, or directed right *lines* in tridimensional space; while q, r, s, are here *not* (generally) rays, or lines, but certain *results* of cardinal analysis, or *instruments* of cardinal synthesis, namely, certain geometrical *quotients* or *factors*, the precise nature of which we have proposed to ourselves to consider more closely soon, but concerning which we have as yet no right to assume that they must necessarily follow, in *all* respects, the same rules of combination among themselves, as the rays a, β, γ. (Compare art. 35).

51. It may be useful here to collect into one *tabular view* (analogous to that of art. 30) the *names* above assigned to the three rays, a, β, γ; which names have been the following:

$$\left.\begin{aligned} a &= \text{Faciend} = \text{Transfaciend}; \\ \beta &= \text{Factum} = \text{Profaciend}; \\ \gamma &= \text{Profactum} = \text{Transfactum}. \end{aligned}\right\}$$

Each of the three *rays*, which are here considered and compared, receives thus, as we see, *two* different *names*, on account of its being regarded in two different *views*, as connected with and concerned in some two out of the three different (although similar) *acts* of faction, profaction, and transfaction; exactly as (in art. 30) each of the three *points*, A, B, C, was formerly tabulated as receiving two names, on account of its connexion with some two of the three acts of vection, provection, and transvection.

52. To draw still more closely together into one common contemplation, or *conspectus*, what has thus been separately shewn in the foregoing and in the present lecture, we may now conceive that the three *rays*, α, β, γ, are three diverging *edges* of a *pyramid*, ABCD, which has a new point, D, for its *vertex*, and for the common origin, or initial point, of the three rays; while the *base* of this pyramid is the *triangle* ABC (of art. 27), which has the three old points, A, B, C, for its three corners. We may then write, in the notation of the former Lecture,

$$\alpha = \mathrm{A} - \mathrm{D}\,; \quad \beta = \mathrm{B} - \mathrm{D}\,; \quad \gamma = \mathrm{C} - \mathrm{D}\,;$$

and shall have also the relations,

$$\left.\begin{aligned} \mathrm{a} &= \mathrm{B} - \mathrm{A} = \beta - \alpha\,; \\ \mathrm{b} &= \mathrm{C} - \mathrm{B} = \gamma - \beta\,; \\ \mathrm{c} &= \mathrm{C} - \mathrm{A} = \gamma - \alpha. \end{aligned}\right\}$$

And we may say that while each of the three *points*, A, B, C, receives two different names, or designations, as belonging at once to *two different sides* of the TRIANGLE OF VECTIONS, ABC, each of the three *rays*, α, β, γ, receives, in like manner, two names, as appertaining at once to *two different faces* of the PYRAMID OF FACTIONS, $\alpha\beta\gamma$; namely, to some two out of the three faces which may be called, respectively, the *face of faction* ($\alpha\beta$ or ADB); the *face of profaction* ($\beta\gamma$ or BDC); and the *face of transfaction* ($\alpha\gamma$ or ADC).

53. All this may be illustrated by the two following diagrams; of which one (fig. 6) is designed to represent the *triangle of vections*, ABC, while the other (fig. 7) is intended to picture the *pyramid of factions*, $\alpha\beta\gamma$.

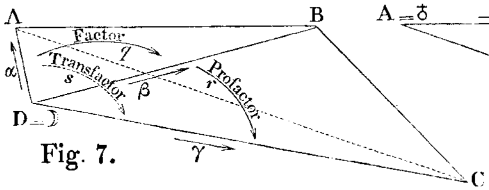


Fig. 7.

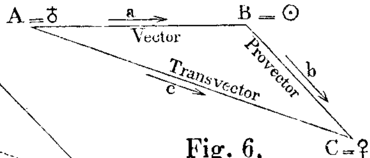


Fig. 6.

In astronomy we may still conceive, as before, that the three points A, B, C, are situated at the centres of the Earth, Sun, and Venus, respectively; and may then imagine that the fourth point, D, is situated at the centre of the Moon.

Thus the three diverging edges of the pyramid, or the three rays, α, β, γ, will coincide, in this astronomical example, with the *selenocentric vectors* of the Earth, the Sun, and Venus, or with the three rays from the centre of the Moon to the centres of those three other bodies.

54. And as (in art. 36) we saw that what we had begun by regarding, in the former Lecture, as the *primary significations* of the marks + and − in geometry, conducted to certain *secondary significations* of *those* two characteristics of operation; so now, from what have been, in the present Lecture, conceived as the *primary* significations of the marks × and ÷, we may observe that we are conducted to certain analogous and *secondary* significations of *these* two *other* marks or characteristics. From expressions of the forms, "*line* PLUS *point*," and "*point* MINUS *point*," we were before led on to the expressions of the forms, "*line plus line*," and "*line minus line*." And, in like manner, from expressions of the forms, "*factor* INTO *ray*," and "*ray* BY *ray*" (where the *rays* do not differ in *kind* from the *lines* before considered, and where the words *into* and *by* are equivalent to the marks × and ÷), we have since been conducted to expressions of the forms "factor *into* factor," and "factor *by* factor;" for we have been led to assert that "Profactor, multiplied into Factor, equals Transfactor" (art. 49), and that "Transfactor, divided by Factor, equals Profactor" (art. 50). It is true that these two last assertions, like the two corresponding enunciations of the preceding Lecture, namely, "Provector *plus* Vector = Transvector" (art. 31), and "Transvector *minus* Vector = Provector" (art. 34), have, *at first*, offered themselves to our notice as mere *abbreviations* of certain other and longer statements, in which the marks + − × ÷ had all retained what we have regarded as their primary significations. But as we saw (in art. 36), that the abridged expressions of the forms "line + line," and "line − line," might suggest a certain derivative or *secondary ordinal synthesis*, and a corresponding derivative or *secondary ordinal analysis*, which might be called

(as in fact they often are called) "*addition and subtraction of lines*," and might be *interpreted* (as in fact they often are interpreted), as answering to the *composition and decomposition of vections* (or of motions); so we may now see that the newer abbreviated expressions of the forms "factor × factor" and "factor ÷ factor," may *suggest* a certain derivative or SECONDARY CARDINAL SYNTHESIS, and a certain other and correspondent derivative or SECONDARY CARDINAL ANALYSIS, which may be *called* "*Multiplication and Division of Factors*," and which admit of being *interpreted* as answering to the COMPOSITION AND DECOMPOSITION OF FACTIONS, or of *operations* of the factor kind.

55. Thus, when (see fig. 6) we assert that the Provector, C − B, from the Sun to Venus, being *added* geometrically to the Vector, B − A, which extends from the Earth to the Sun, gives, as the geometrical SUM, the Transvector, C − A, which goes from the Earth to Venus; we may INTERPRET the assertion (whatever the original *motives* for enunciating it may have been), as expressing that *to go straight accoss* (*trans-*) from the earth to the planet, if we attend only to *the total or final* EFFECT of this process, or to the ultimate *change of position* accomplished by this mode of transport, *comes to the same thing*, as to go *first* from the Earth to the Sun, and *afterwards* from the sun to the planet. And in like manner when we assert (see fig. 7), that the Profactor, $\gamma \div \beta$, being multiplied geometrically *into* the Factor, $\beta \div \alpha$, produces the Transfactor, $\gamma \div \alpha$, we may *interpret* the assertion by saying that to change at once the selenocentric ray or vector of the Earth to the selenocentric vector of Venus, is, *as to final effect*, the *same thing*, as to change *first* that selenocentric vector of the Earth to the selenocentric vector of the Sun, and *afterwards* to change this selenocentric vector of the Sun to the selenocentric vector of the Planet. An *act of vection* may be compounded with a *subsequent act* of *pro*-vection into one *single* act of *trans*-vection; and, in like manner, an *act of faction* (which changes one ray or vector to another) may be *compounded* with an act of *pro*-faction following it, into one *single* act of *trans*-faction, which as to its *effect*, or the ultimate *result* of its operation, shall be equivalent to the system of those two former acts of the same kind. To move successively *along the two sides*,

AB, BC, of any triangle, ABC, is to move, *upon the whole, from* the first point, A, *to* the last point, C, of the *base*, AC. To sweep over the face, ADC, of the pyramid, ABCD, from the edge DA, to the edge DC, or from the ray α to the ray γ, is an operation which has the *same first subject*, and the *same last result*, as to sweep *first* over the face, ADB, from the edge DA to the edge DB, or from the ray α to the ray β, and *then* over the face BDC, from the edge DB to the edge DC, or from the ray β to the ray γ. (Compare the commencement of art. 48.)

56. It has been noticed (in art. 54) that there exist two kinds of *secondary analysis*, ordinal and cardinal, which answer to the two kinds, recently illustrated, of secondary synthesis: namely, those two modes of analysis which consist, respectively, in the *decomposition* of *vections*, and of *factions*. The first or ordinal kind of secondary analysis has been called the *subtraction of lines;* the second or cardinal kind of secondary analysis has been called the *division of factors*. The diagrams lately exhibited (figures 6 and 7) may serve to illustrate these two processes. Thus we have been led to say (see fig. 6), that the *subtraction* of the Vector B – A, from the Transvector C – A, gives the Provector C – B as the *remainder;* or that the subtraction (compare art. 34) of the geocentric vector of the Sun from the geocentric vector of Venus, leaves, as remainder, the heliocentric vector of the planet. And whatever *motive* of abridgment may have *originally led us* to enunciate this assertion, while the mark – was still confined by us to what we regarded as its primary signification, we may now be led to INTERPRET the assertion as expressing, that if the act or process of *transvection*, from the earth A to the planet C, be DECOMPOSED into *two* successive vections, of which the *first* is the *given* act of vection from the earth to the sun B, then the *second* component must be (or be equivalent to) the act of *provection*, from the Sun B to Venus C. This, then, is an *example* of what we have called *secondary ordinal analysis*, or ANALYSIS OF VECTION, arising out of that *primary* and ordinal analysis, or ANALYSIS OF POSITION, namely, the examination or study of the position of one *point* B as compared with another point A, which primary sort of analysis in geometry was considered in the former Lecture. And in like manner, from that primary and

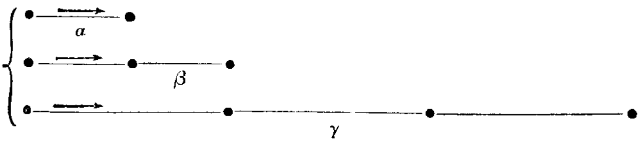
cardinal analysis, or ANALYSIS OF DIRECTED DISTANCE, on which, in the present Lecture, we have entered, by comparing one *ray* β with another ray α, we have been conducted to a *secondary cardinal analysis*, or to an ANALYSIS OF FACTION; that is, to a *decomposition of one* FACTOR ACT *into two other acts of the same kind*, which may be illustrated by figure 7. For we may say that if the act or process of *transfaction*, from the ray α to the ray γ, that is (in our example) from the selenocentric vector of the earth to the selenocentric vector of the planet, be *decomposed into two successive acts* of the same kind, of which the *first* is *given* to be that act of *faction* whereby we pass from the ray α to the ray β, or from the selenocentric vector of the earth to that of the sun, then the *second* is *found* to be (or to be equivalent to) that *other* act, of *profaction*, whereby a passage of the same sort is made (along the remaining face of the pyramid) from the ray β to the ray γ, or from the selenocentric vector of the Sun to the selenocentric vector of Venus. And thus we may, if we think fit, INTERPRET the assertion, that "the Transfactor divided by the Factor gives the Profactor as the Quotient;" or in symbols, we may *interpret* thus the formula,

$$\gamma \div \beta = (\gamma \div \alpha) \div (\beta \div \alpha);$$

whatever desire of such *abbreviation* as might be gained by the omission of the *twice-recurring signs*, $\times\, \alpha$, or by the suppression of the *twice-repeated words*, "Multiplied into Faciend," may have *first induced* us to adopt the latter usual formula, or the former mode of verbal enunciation, while the *mark* $\div$ and the *name* Division were still, as yet, *confined* by us to what we regarded as their *primary* significations: and were therefore employed to denote only the COMPARISON OF ONE DIRECTED DISTANCE WITH ANOTHER.

57. As *examples* of such comparison or analysis, which may illustrate what has been already said, we shall here consider a few very simple cases; in *some* of which the compared rays shall *agree* with each other in *direction*, but *differ* from each other in *length*; while in *other* cases they shall, on the contrary, agree in length, but differ in direction.

Supposing then, first, that we have not only (as in the example of article 39), $\beta = a + a$, but also $\gamma = \beta + \beta + \beta$; as is represented in this figure,

Fig. 8.


We shall then evidently have, not only $\beta \div a = 2$ (as in 39), but also $\gamma \div \beta = 3$, and $\gamma \div a = 6$. In this case, then, the factor q, the profactor r, and the transfactor s, are respectively equal to the cardinal numbers, 2, 3, 6; and the general relation (of art. 49) connecting them, or the formula, $s = r \times q$, becoming here simply $6 = 3 \times 2$, is obviously, in this example, consistent with ordinary arithmetic; as is also the inverse formula (of art. 50), $r = s \div q$, since it becomes here $3 = 6 \div 2$. Now (compare art. 40), that *division of the ray*, γ, or of the line $\beta + \beta + \beta$, or of $6 \times a$, *by the ray* or line β, or $2 \times a$, which conducts to the quotient 3, is what I call a *primary cardinal analysis*, or is an example of what I regard as the *primary* operation of Division in Geometry; since it leads to an expression for the *relative length* of a line γ, as compared with another line β; the *relation of directions* being already known to be, in the present case, a relation of *sameness*, or identity. And on the other hand the division of the *number* 6 by the *number* 2 is an example of what I call a *secondary cardinal analysis*; at least when this operation is regarded as being the comparatively abstract *analysis of the act of sextupling*, whereby that act (of *transfaction*) is here decomposed into the *given act of doubling* (which is in this case the act of *faction*), and another act of the same sort (the act of *profaction*), which is here *found*, by this decomposition, to be the *act of tripling*, as is expressed by the arithmetical formula $6 \div 2 = 3$, according to the mode of interpretation of such formulæ which has been above proposed (in art. 56). In like manner in the *synthetic aspect* of the question, or of the lines and numbers here compared and combined, I regard as *primary* that *cardinal synthesis* by which we *construct the ray* γ, or the line $\beta + \beta + \beta$, by *operating on another ray* β with the number 3 as a multiplier; and I regard as *secondary* that other sort of cardinal synthesis, by which we *produce the number* 6 (the transfactor), by multiplying a number 2 (the factor), by another number 3 (the profactor); or by *compounding the two successive acts* of doubling and of tripling, *into a third act* of the same sort, namely, the act of sextupling, as is expressed, according to the mode of interpretation above proposed (in art. 55), by writing $6 = 3 \times 2$. We may, however, according to another mode of interpretation already mentioned (in 49 and 50), *retain the formulæ* $6 = 3 \times 2$, and $6 \div 2 = 3$, *without introducing the conceptions* of such composition and decomposition of factions, provided that we regard these formulæ as *abbreviations* for the fuller assertions

$$6 \times a = 3 \times 2 \times a, \text{ and } (6 \times a) \div (2 \times a) = 3,$$

in which the signs $\times$ and $\div$ are used in what we have called their *primary* significations in geometry. And similarly in other cases, where the lengths *only*, but *not* the directions, of the rays a, β, γ, are different; and when therefore the factor, profactor, and transfactor, are ordinary *numbers*, which, in *this* class of cases, are always *positive* or *absolute*, although they may become fractional or incommensurable.

58. A slightly different class of cases may here be usefully noticed, as conducting, on the same general plan, to the consideration of *negative numbers;* and as reproducing the usual rules for the multiplication and division of such numbers: while it will also serve as an useful preparation for those more complex products and quotients, of which we shall afterwards have to speak.

By principles already laid down, the *sum* of any two *opposite* lines is a *null* or evanescent line; for the transvector C − A vanishes, when the provectum C, becoming a revectum, coincides with the vehend A. In fact it is evident that if we first *go*, along any line AB, from A to B, and then *return* along the same line, from B to A, we occupy the *same final position* as if we had *not moved* at all. We may then say that

"REVECTOR + VECTOR = ZERO;"

and that conversely,

"REVECTOR = ZERO − VECTOR;"

the word *zero*, or the symbol 0, being understood to denote a *null line*, when used in such connexions as these. Thus

$$(A - B) + (B - A) = 0;$$

and

$$(A - B) = 0 - (B - A);$$

which latter equation may be *abridged* to the following formula (familiar in ordinary algebra):

$$A - B = -(B - A);$$

while, by a similar abridgment of discourse, we may say, in words, that

REVECTOR = MINUS VECTOR:

understanding or tacitly supplying the word *zero* before the word *minus*, in order to bring this mode of expression into harmony with others which have been already discussed. In like manner, if we conceive the provectum C to coincide with the provehend B (and not now with the vehend A), it will be the provector $C - B$ (instead of the transvector $C - A$), which will vanish, while the transvectum and vectum will coincide; we shall, therefore, have the enunciation:

VECTOR = ZERO + VECTOR;

which may be *abridged* to the following form:

VECTOR = PLUS VECTOR;

the word *zero* being still *understood*. In symbols we have (as in algebra),

$$B - A = (B - B) + (B - A) = 0 + (B - A);$$

and more concisely, *omitting* the 0,

$$B - A = +(B - A).$$

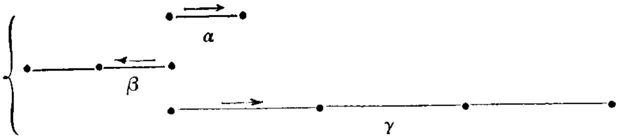
Thus, a being a symbol for a ray, or for a vector, $+a$ comes to be *another symbol* for the *same ray* or vector; and $-a$ comes to be a symbol for the *opposite ray*, or for the revector corresponding. In like manner, after agreeing that $2a$ shall denote concisely the same thing as $2 \times a$, the symbols $+2a$ and $-2a$ come to denote, respectively (as in fact they are often employed to do), the double of the ray *a itself*, and the *opposite* of that doubled ray; and similarly in other instances.

59. Now, I think, that the clearest way of viewing *positive* and *negative numbers*, at least as connected with Geometry (for I endeavoured many years ago to shew that such numbers might

be regarded as presenting themselves in Algebra, according to the view which I took of that science, as *results of the division of one step in time by another*), is to regard such numbers as being each the QUOTIENT of the division of one *step in space*, that is, of one ray or vector, *by another step* in space, which has its direction either *exactly similar* or else *exactly opposite* to the former. Thus, the cardinal numbers, "positive two" and "negative two," or $+2$ and -2, would offer themselves in this view as certain geometrical quotients, or at least as quotients of certain geometrical divisions, of that general kind which has been considered in the present Lecture, namely, as quotients of the forms,

$$+2 = +2a \div a; \quad -2 = -2a \div a;$$

where the symbols $+2a$ and $-2a$ are interpreted as in the foregoing article, and do *not* (here) denote *abstract numbers*, but certain comparatively *concrete conceptions*, namely, certain *rays*, or *lines*, or *steps* in space. Observe now this diagram,

Fig. 9.


which is designed to picture the conceptions of the relations, $\beta = -2a$, $\gamma = +6a$; and you will see that for *this* set of rays, a, β, γ, the values of the factor, profactor, and transfactor, are the following negative or positive numbers:

$$\begin{cases} \text{Factor} = q = \beta \div a = -2; \\ \text{Profactor} = r = \gamma \div \beta = -3; \\ \text{Transfactor} = s = \gamma \div a = +6. \end{cases}$$

You see, then, that the general formula or *rule of multiplication* assigned in the present Lecture, namely, the rule

$$\text{Transfactor} = \text{Profactor} \times \text{Factor},$$

gives here, again, as in art. 57, a result agreeing with received principles, namely, with those of elementary algebra, since it gives

$$(+6) = (-3) \times (-2);$$

or in words, the result, that "Positive Six equals the product of Negative Three into Negative Two." You see, too, that (in consistency with our present views) we may *either* regard this elementary result as a mere *abbreviation* of the formula

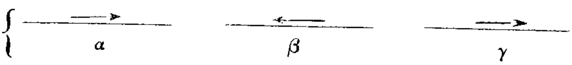
$$(+6) \times \alpha = (-3) \times (-2) \times \alpha,$$

where the sign $\times$ may still be considered as being used in what we have called its *primary* sense; or we may *interpret* the same result of multiplication, of the two negative numbers proposed, as signifying that the *two successive acts*, of *negatively doubling* and *negatively tripling*, *compound* themselves into the *single* act of *positively sextupling*. And it is obvious that analogous remarks apply to the converse formula of division,

$$(+6) \div (-2) = (-3).$$

In general, this way of considering the multiplication and division of positive or negative numbers (whether whole or fractional or incommensurable), reproduces the usual *rule of the signs*, and is, in all its consequences, consistent with common algebra.

60. A few words may, however, be said here upon the RULE OF THE SIGNS just referred to, in the hope that they may make that *rule* and the present *principles* throw light upon each other. Suppose, then, that we have, as in this figure,

Fig. 10.


the relations $\beta = -\alpha$, $\gamma = -\beta$, which give also (as the figure shews) the relation $\gamma = +\alpha$. We might express these relations under the forms

$$\beta = (-1) \times \alpha, \gamma = (-1) \times \beta, \gamma = (+1) \times \alpha,$$

and so arrive, on the plan of the foregoing article, at the well-known equation of algebra,

$$(-1) \times (-1) = (+1).$$

But we might *also* write

$$\beta = (-) \times \alpha, \gamma = (-) \times \beta, \gamma = (+) \times \alpha;$$

regarding the signs $(+)$ and $(-)$, when *thus* employed, as being *themselves* of the nature of geometrical *factors* or *multipliers*;

because if they operate at all, they do so on the DIRECTIONS of the rays, or lines, or steps, to the symbols of which they are prefixed, with the MARK OF FACTION $\times$ interposed; so that their operation, whether non-effective or effective, comes to be included under that general head or class of operation to which it has been already stated that we apply the name *multiplication* in geometry. And then the general relation of multiplication to *division*, or of $\times$ to $\div$, will enable us to form also, as expressions of the *same* relations between the three rays α, β, γ, in fig. 10, combined with the nomenclature of preceding articles, the following little table:

$$\begin{cases} \text{Factor} = q = \beta \div \alpha = (-)\,; \\ \text{Profactor} = r = \gamma \div \beta = (-)\,; \\ \text{Transfactor} = s = \gamma \div \alpha = (+). \end{cases}$$

The general formula "profactor *into* factor equals transfactor," or $r \times q = s$, becomes, therefore, here, the particular formula,

$$(-) \times (-) = (+);$$

and the converse general formula, "transfactor *by* factor equals profactor," or $s \div q = r$, becomes here,

$$(+) \div (-) = (-).$$

The effect of the sign $(-)$, when *thus* used as a factor, being to *invert the direction* of the ray or step on which it operates (as is exhibited by the *arrows* in the figure), this factor $(-)$ itself may be said to be an INVERSOR; whereas the *other* sign $(+)$, when similarly used as a factor, may be called, by contrast, a NON-VERSOR, because *its* effect is simply to *preserve* the direction of the ray or step on which *it* operates, or *seems* to operate. We may also say (by the introduction of another new but convenient term), that the sign $(+)$, as a factor, NON-VERTS the ray, to the symbol of which it is prefixed; or that its effect is a *non-version:* whereas the sign $(-)$, as before, *in*-verts, or its effect is an *inversion.* And thus the formula

$$(-) \times (-) = (+)$$

may (on our general plan) be interpreted as expressing the result of a certain composition of factions; that is, *here*, a composition of *versions*, or still more precisely, a *composition of two successive inversions*, into a single equivalent operation, namely, a *non*-version. It signifies, when translated into ordinary words, that if we *twice* successively invert, or *reverse*, the direction of any step, we do what is, *upon the whole*, equivalent to leaving the step *unchanged:* since, by this *double* alteration, we *recover*, or restore, the original direction of that step. And in like manner the converse formula,

$$(+) \div (-) = (-),$$

may, on the same plan, be interpreted as expressing the *decomposition of a non-version* into two successive inversions; or as signifying that if it be required to follow up a first inversion of a step by some *second* operation, which shall, upon the whole, produce the effect of a non-version, or shall *restore* the step to the direction which it originally had, this second or successive operation must be *itself* an inversion, or some operation equivalent thereto. Remarks precisely similar apply to all the other formulæ of this kind, such as

$$(+) \times (-) = (-), \quad (-) \div (-) = (+);$$

which may all be in like manner *interpreted*, and with this interpretation *proved*, if they be regarded as relating to compositions and decompositions of inversions and nonversions of a *ray*, or more generally of a STEP in *any* proposed progression: the general rule being evidently that any *even* number of *in*-versions are equivalent, on the whole, to a *non*-version; and that, therefore, any *odd* number of inversions are equivalent to a *single* inversion; or produce the same *final* effect, as that single inversion would do.

61. It is evident also that if we should prefer to look at these last signs (+) and (−) in their *analytic* instead of their *synthetic* aspect, or to regard them as *quotients* rather than as *factors*, they would then (on the general plan already mentioned) come to be considered respectively as symbols of the RELATIONS of SIMILARITY and OPPOSITION between the *directions* of any two rays or steps. Thus we might write again the formulæ,

$$\beta \div \alpha = (-), \quad \gamma \div \alpha = (+),$$

in connexion with the lines of fig. 10, in order to express that on *analyzing the directions* of β and γ (as marked by arrows in that figure), considered as *analyzands*, with respect to the direction of a considered as an *analyzer*, we should find by this comparison (which we regard as being still a species of *cardinal analysis*), that the relation of directions between β and a is a relation of *opposition;* but that the relation of directions between γ and a is a relation of *similarity*. And in this analytic aspect of the signs $(+)$ and $(-)$ as certain *cardinal quotients*, the formula $(-) \times (-) = (+)$ may be interpreted as expressing that *two relations* of *opposition* (of directions) *compound* themselves into *one* relation of *similarity;* or that the *opposite of the opposite* of any direction is *the original direction itself:* while analogous and equally simple interpretations might be given for all other formulæ of this sort, on the plan of the present Lecture.

62. In the two foregoing articles the three lines a, β, γ, which were compared among themselves, were supposed to have *equal lengths*, and to differ (so far as they differed at all) in their *directions* only; or at most in their *situations* in space, from which situations, however, we *abstract*, in the present inquiry or contemplation. The only *operators* of the cardinal kind, whether effective or non-effective, which have thus been brought into view by the consideration of the example of art. 60, have been (as we have seen) the *factors* $(+)$ and $(-)$, regarded as signs or characteristics of *nonversion* and of *inversion* respectively; and *not* (when used in *this* sort of connexion) as marks of *addition* and *subtraction;* although it was shewn (in articles 58, &c.) how, in the *progress of* NOTATION those earlier significations of $+$ and $-$ which were connected with addition and subtraction, *might* gradually come to suggest or to permit that *other* use of them, whereby they are connected with multiplication and division.

63. On the other hand, in the example of art. 57, the three lines a, β, γ, which were *there* compared, had all the *same direction*, and *differed* only in their *lengths*. In *that* example, therefore, we had not occasion to consider any kind of *turning*, or of VERSION; but we had, on the contrary, occasion to consider what may be called a *stretching*, or a TENSION, namely, that *other* operation of the factor kind, by which we pass from one given length (and not from one given direction) to another. It was on

extension (not on direction) in space, that we operated in that earlier example; the act performed was an act of a *metric*, and not one of a *graphic* character. The *agents*, therefore, or the *factors*, in those earlier operations of the cardinal kind which were considered in art. 57, may naturally, in consistency with the plan of nomenclature employed in these Lectures, receive the general name of TENSORS; and we may say, more particularly, that the factor, profactor, and transfactor, were (in the example here referred to) a *tensor*, *protensor*, and *transtensor* respectively. And although these three tensors, in the example of art. 57, being the three cardinal numbers 2, 3, and 6 respectively, were thus each greater than the number *one*, and so had the effect of actually *lengthening* the line (α or β) on which they operated; yet it seems convenient to enlarge by definition the signification of the new word *tensor*, so as to render it capable of including also those other cases in which we operate on a line by *diminishing* instead of *increasing* its length; and generally by altering that length in any definite *ratio*. We shall thus (as was hinted at the end of the article in question) have fractional and even incommensurable *tensors*, which will simply be numerical multipliers, and will all be *positive* or (to speak more properly) SIGNLESS NUMBERS, that is, unclothed with the algebraical signs of positive and negative; because, in the operation *here* considered, we abstract from the directions (as well as from the situations) of the lines which are compared or operated on. Thus the three acts, of doubling a line, of halving it, and of changing it from the length of a side to the length of a diagonal of a square, shall be regarded as being, all three, *acts of tension*; the tensors in these three respective acts being the integral number 2, the fractional number $\frac{1}{2}$, and the incommensurable number $\sqrt{2}$. The act of *restoring* a line to its original length, after that length had been altered by a previous act of tension, might be called an act of RE-TENSION, and the agent in the second operation might be called a RE-TENSOR (compare art. 44); thus any tensor and its answering retensor would simply be two numbers of which each is (what is commonly called) the *reciprocal* of the other; or, in their analytic aspect, they would represent ratios mutually *inverse*. The number 1 might be called a NON-TENSOR, because it makes no actual alteration in the length of the line which it multiplies; just as the sign (+) was lately called a NON-VERSOR, because it leaves unchanged the direction on which it seems to operate. And the general formula for the multiplication of such *signless numbers*, or for the composition of ratios of lengths (or other magnitudes), will offer itself with these conceptions and denominations, as a particular *case* of the general multiplication of *factors*, or of the composition of cardinal relations, under the form (compare art. 49):

$$\text{TRANSTENSOR} = \text{PROTENSOR} \times \text{TENSOR};$$

together with the converse formula of division (compare art. 50):

$$\text{PROTENSOR} = \text{TRANSTENSOR} \div \text{TENSOR}.$$

64. As regards the example of art. 59, each act of faction *there* considered may be said to have been *compounded* of an act of tension, and an act of inversion or of nonversion, according as the numerical (but *not signless*) multiplier employed was a negative or a positive number; and we may express this conception by writing, in reference to that example:

$$(-2) = (-) \times 2; \quad (+6) = (+) \times 6;$$

with analogous expressions for all other positive or negative numbers. It is also evidently allowed to write, with a different arrangement of the factors,

$$(-2) = 2 \times (-); \quad (+6) = 6 \times (+);$$

since it comes (for example) to the same thing, whether we first double a step and afterwards reverse its direction, or first reverse and afterwards double. We may agree to give the general name of SCALARS to all positive and negative numbers (that is to the REALS of ordinary algebra), on account of the possibility of conceiving all such multipliers to be represented, or laid down, on one common but indefinite *scale*, extending from $-\infty$ to $+\infty$, that is, from negative to positive infinity.

65. Proceeding now to a more general examination of the *directions* of lines, or rays, in *space*, let us consider a somewhat more complex case of the (analytic) comparison of such directions, or of the (synthetic) composition of versions, than any of those

which were discussed in recent articles: and for this purpose let i, j, k, denote three straight lines *equally long*, but differently directed; let it be also supposed that these three different directions are *rectangular* each to each; and to fix the conceptions still more precisely, let us conceive that these directions of i, j, k, are respectively *southward*, *westward*, and *upward* (in the present or in some other part of the northern hemisphere of the earth); so that i and j are both horizontal, but k is a vertical line. We may further imagine that the common length of these three lines is equal to some assumed *unit* of length, or more particularly, that it is a *foot;* so that i is or denotes a southward foot, j is a westward foot, and k is an upward foot. Then (by art. 58) + i, + j, + k, will be other symbols for the same three directed lines; but − i, − j, − k, will denote respectively a northward, an eastward, and a downward foot. This being agreed upon, let the three diverging edges, α, β, γ, of the pyramid in fig. 7 (of art. 53), be conceived to be each a foot long, and to be directed respectively towards the northern point of the horizon, the zenith, and the east point, so that we may write the equations:

$$\alpha = -\mathrm{i}, \quad \beta = +\mathrm{k}, \quad \gamma = -\mathrm{j}.$$

The *pyramid* being thus constructed, we may next proceed to study the three separate *acts* of faction, profaction, and transfaction, by which we may pass respectively from α to β, from β to γ, and from α to γ, by operating on the *directions* of the rays or lines α and β, and, therefore, by performing what may be called acts of VERSION, PROVERSION, and TRANSVERSION: since it is clear that there is, in the present case, no act of *tension* performed, the three lines which are compared being supposed to be all equally *long*. The *agents* in the three acts which we are thus to study, may be called respectively the VERSOR, the PROVERSOR, and the TRANSVERSOR; and we may already enunciate, as a particular case of the general formula of *multiplication of factors* in art. 49, the relation:

$$\text{TRANSVERSOR} = \text{PROVERSOR} \times \text{VERSOR};$$

which must, by the general conceptions and definitions of multiplication already stated, hold good for *every composition of ver-*

sions. We may also, in like manner, as a particular case of the general formula of *division of factors* in art. 50, enunciate this converse relation,

PROVERSOR = TRANSVERSOR ÷ VERSOR ;

which is to be regarded as being likewise valid, by the *general* significations of the terms employed, for *every* case of *decomposition of versions*, or of rotations in geometry. We may also modify the phraseology of former articles, respecting the three lines α, β, γ, themselves, considered now as the subjects or the results of operations of the *versor* kind, by *naming* those three lines as follows (compare the table in art. 51) :

$$\begin{cases} \alpha = \text{Vertend} = \text{Transvertend}; \\ \beta = \text{Versum} = \text{Provertend}; \\ \gamma = \text{Proversum} = \text{Transversum}; \end{cases}$$

in order to mark, by this nomenclature, that we now abstract from the lengths of the lines, or that we treat those three lengths as equal. We shall thus be able to assert generally (compare art. 41), that

VERSOR × VERTEND = VERSUM,

and that

VERSUM ÷ VERTEND = VERSOR ;

with other analogous formulæ (compare articles 47, 48) for proversion and transversion respectively. But *what* the *particular* acts of version *are*, for any particular set of lines or rays, as (for example) for the set mentioned at the beginning of the present article, it still remains to consider.

66. In this consideration or inquiry, we may assist ourselves by remembering the general remarks which were offered at an earlier stage of the present Lecture (in articles 39 and 40). The *lengths* of the lines which are to be compared being (in the present question) *equal* to each other, the *metric* element of the inquiry disappears, and only the *graphic* element remains. We have, therefore, only now to inquire, as concerns the lines α and β, through *what angle*, in *what plane*, and towards *which hand*, are we to *turn* the line α as a given *vertend*, in order to make it

attain the proposed direction of the *versum*, that is of the line β? For the *answer* to this inquiry, when it shall be, in any manner, with sufficient clearness and fulness assigned, will be, under one form or other of expression, a sufficient description, statement, or particularization of the sought *versor*, which we have already, by anticipation, *denoted* by the symbol $\beta \div a$, and have called a cardinal *quotient*.

67. Now, with the particular directions above assumed or assigned, for the vertend and versum, or for the lines a and β, namely, those otherwise denoted (in 65) by $-$i and $+$k, or the (horizontally) northward and the (vertically) upward directions, it is clear that the *angle* of version is a *right* angle; the *plane* is *meridional;* and the *axis* of *right handed* rotation, from a to β, is a right line directed *westward*. In that little model of a transit instrument which you see here, the line a may be conceived to be the telescope when pointed to a north meridian mark; and β the same telescope, directed towards the zenith. And when I lay my hand on the westward half of the axis in the model, and turn that part *right handedly*, with a motion of the *screwing* kind, you see that the *northern* (or *object*) end of the telescope comes to be *elevated*, while the *southern* (or *eye*) end is *depressed*. Continuing this motion of rotation through a quadrant of altitude, you see that I have *erected* the telescope in the model, in such a manner as to cause it to attain a vertically upward direction; and that thus I *have*, in fact, changed the telescope (that is, its *object half*) *from* the direction symbolized by a *to* the direction symbolized by β. The required act of version, symbolized by $\beta \div a$, has, therefore, in this case, been actually and practically performed.

68. And since the (mechanical) *agent* in producing this (mechanical) rotation, or in this right-handed (or *screwing*) act of version, has been an *axis* or handle directed to the *west*, which direction has also been lately supposed (in art. 65) to belong to the line denoted by the symbol $+$j, I propose now to denote the *versor* itself, or the CONCEIVED AGENT of the *conceived version*, or of the purely *geometrical* rotation from a to β, by the *connected symbol j;* availing myself (as you see) of the distinction between the roman and the italic alphabets, to mark, at least temporarily,

the distinction between the two different conceptions of a line, as a *turned* and as a *turning* thing; a versum and a versor; a subject of operation and an operator. We shall thus have, on the general plan of notation already stated or sketched for you, the formulæ:

$$\beta \div a = (+\mathrm{k}) \div (-\mathrm{i}) = j;$$
$$j \times a = j \times (-\mathrm{i}) = \beta = +\mathrm{k};$$

and the "*j-operation*," or the operation of multiplying a line by the factor or versor j, is seen to have the effect of elevating a transit telescope from that position in which it is directed to the north point of the horizon, to that other position in which it is directed towards the zenith. The conception of this operation may be illustrated by figure 11, where the *axis* j is drawn as directed to the west, and as *ready* to operate on the telescope or line a, which line is, *before* the operation, represented as directed towards the north; but is to be conceived as taking, *after* that operation, the direction towards the zenith, represented by β in fig. 12: with which two figures, I shall here, by anticipation, associate a third (fig. 13).

Fig. 11.

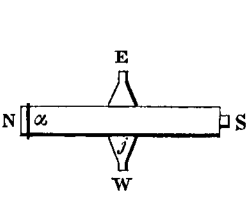


Fig. 12.

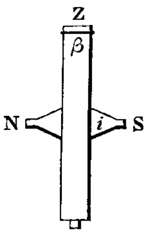
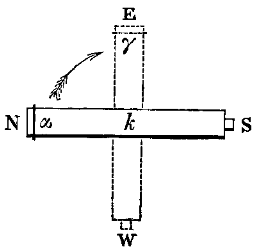


Fig. 13.



69. Having thus passed, by the way of rotation, from a to β, or from $-\mathrm{i}$ to $+\mathrm{k}$, there is no difficulty in passing similarly from β to γ, or from $+\mathrm{k}$ to $-\mathrm{j}$. The act of *version* having been studied and symbolized, it becomes easy to study and symbolize, in like manner, the subsequent but analogous act of *proversion*. We have passed from a northward to an upward position of the telescope; and we are now to pass from an upward to an eastward position thereof. This cannot, indeed, be done by any such *meridional* motion as belongs to an *ordinary* transit telescope;

but it can be done by that *other* important mode of motion of a telescope, of the *extra*-meridional kind, in the plane of the *prime-vertical*, which has been used, with great success, in some celebrated geodetic surveys, and also at some fixed observatories, in Russia and elsewhere. Having already erected the telescope to the zenith in this little model of a transit, you see that I can turn the model through a quadrant of azimuth, so as to cause that axis, or *semiaxis*, which had been directed *westward*, to take the *southward* direction. And if I *now* lay my hand on the same physical or mechanical semiaxis as before, but in its *new* and southward direction, you see that the *same* sort of *screwing* motion, as that which was before employed, being continued through the same *angular* quantity, namely, through a *quadrant* of rotation of the telescope, in the plane of the *prime vertical*, has the effect of turning that telescope from the upward to the eastward direction, or from the direction of β to that of γ, that is, from the direction of $+\mathrm{k}$ to that of $-\mathrm{j}$. In short, you see that the required act of Proversion is thus effected; and that I may naturally denote the *Proversor*, or the *agent* of the proversion, on the plan of the foregoing article, by the symbol i; because, as you may see illustrated by the diagram last referred to (fig. 12), the axis, or handle, of this proversion, is, like the line already denoted by $+\mathrm{i}$, a line directed towards the south. We are thus led to write the equations:

$$\gamma \div \beta = (-\mathrm{j}) \div (+\mathrm{k}) = i;$$
$$i \times \beta = i \times (+\mathrm{k}) = \gamma = -\mathrm{j};$$

by *combining* which with the equations of the foregoing article, on the plan of art. 49, we obtain these other formulæ:

$$i \times j \times a = \gamma; \quad i \times j = \gamma \div a.$$

70. Proceeding to consider the *transversion*, we are next to inquire what *one* rotation in a *single plane* would bring the vertend a into the direction of the proversum γ; or would cause the telescope to pass, by a *single* act of turning, from its original and northward, to its final and eastward direction. And it is clear, either from the model before you of the eight-feet Circle, which

belongs to the Observatory of this University, or from the little diagram above drawn (fig. 13), that the plane of this transversion is *horizontal;* that its angular quantity is a *quadrant;* and that, if the rotation be still conceived to be *right*-handed, its axis is a line directed vertically *upwards:* so that the *Transversor* itself may be denoted (on the plan of recent articles) by the italic letter k, because the axis or handle of its operation has the direction of the line which we have above denoted by + k. We shall thus have the formulæ :

$$\gamma \div a = (-\mathrm{j}) \div (-\mathrm{i}) = k\ ;$$
$$k \times a = k \times (-\mathrm{i}) = -\mathrm{j}.$$

And by comparison of the last value of $\gamma \div a$, with that assigned in the preceding article, or by the general principle that transversor = proversor × versor (art. 65), we arrive at the simple but useful equation following :

$$i \times j = k\ ;$$

which may either be interpreted (synthetically) as asserting that the quadrantal rotation j round a westward axis, being succeeded by another quadrantal rotation i, round a southward axis, produces finally, and upon the whole, the same change of direction as that third quadrantal rotation k would do, which is performed round an upward axis, these three rotations being all supposed to be right-handed; or (analytically) as expressing a *composition of relations of directions* in space, which *corresponds* to this *composition of rotations.*

71. After settling, as above, the significations of the symbols i, j, k, regarded as certain *quadrantal versors,* or as symbols denoting the conceived agents or *operators* of certain quadrantal and right-handed rotations in the three rectangular planes of the prime vertical, the meridian, and the horizon, round axes directed respectively towards the south, the west, and the zenith ; we may proceed to investigate, on similar principles, and by analogous compositions of rotations, the symbolic values of all the *other binary products* of these three factors or versors i, j, k; and should find for *each* such product a DETERMINATE result, unaffected by any change of the line (a) assumed as the original *vertend,*

which change the general plan of the construction might allow. Thus, in order to find *anew* the value of the product $i \times j$, we may indeed vary the vertend α, since we *need not* assume this line to be (as was supposed in art. 65) a *foot* directed towards the north. We might assume the line α to denote any *longer* or *shorter* line in the same northward direction; but then we should only alter, in the *same ratio*, the lengths of the two other lines β and γ, without their ceasing to be directed respectively towards the zenith, and the east, so that the geometrical quotient $\gamma \div \alpha$, or the product $i \times j$, would still be found equal to k, since the proversum γ would still be a line of the same length as the vertend α, and would still be advanced beyond it by a quadrant of azimuth, while both these lines would still be contained in the same horizontal plane, if they be conceived to radiate from one common origin. We might even assume the vertend α to be a line directed to the *south*, and not to the north as before; for the only effect of this change would be that the versum β would take a *downward* (instead of an upward) direction; and that the proversum γ would be directed to the *west*, instead of being pointed to the east: and on finally comparing the (new) westward direction of γ with the (new) southward direction of α, we should find that γ was *still*, as before, more advanced in azimuth than α by a quadrant, both being still in a horizontal plane, so that $\gamma \div \alpha$ would still be found equal to k. It was thus (for example), that in the recent act of *version* (68), the *eye-end* of the telescope in the model was *depressed* from the south to the nadir; while in the *proversion* (69), the same eye-end was *elevated* from the nadir to the west: and the *same* horizontal *transversion* (70), which brought the *object*-end from north to east, brought *also*, at the same time, the *eye*-end from south to west. In symbols, retaining the recent significations of i, j, k, as well as those of i, j, k, we might have assumed,

$$\alpha = +\mathrm{i}, \quad \beta = -\mathrm{k}, \quad \gamma = +\mathrm{j},$$

instead of the values or directions which were assumed for α, β, γ, in art. 65; and then we should have had the relations,

$$\beta \div \alpha = (-\mathrm{k}) \div (+\mathrm{i}) = j;$$
$$\gamma \div \beta = (+\mathrm{j}) \div (-\mathrm{k}) = i;$$
$$\gamma \div \alpha = (+\mathrm{j}) \div (+\mathrm{i}) = k;$$

whence there would have followed, as before, the equation,

$$i \times j = k.$$

Nor could any variation of this result be obtained by assuming *other* positions of α; for the plan of construction *requires* that this line α should have *either* a northward *or* a southward direction, if it is to be used as the vertend in the determination of the product $i \times j$; since it is to be in the plane of version, that is here in the meridian plane, and is also to be perpendicular to the versum, or provertend, β; which latter line β must lie at once in the two planes of version and proversion, or in the planes of the meridian and prime vertical, and must, therefore, be a vertical line, directed either upwards or downwards.

72. With respect to the *other* binary products of i, j, k, it is easy to perceive, first, that we have, by an exactly similar composition of rotations, the formulæ,

$$j \times k = i, \text{ and } k \times i = j;$$

which only differ from the formula $i \times j = k$, by a *cyclical permutation* of the symbols, and can, on this account, be easily *remembered.* In fact if it were required to determine directly the value of the product $j \times k$, on the same plan of construction as before, we should have to assume a direction for the versum β, which should be contained at once in the two planes of version and proversion, or be perpendicular at once to the axes of the two successive rotations; thus β must be perpendicular to both k and j, and must, therefore, have one or other of the two opposite directions denoted by the ambiguous symbol $\pm$ i; and by a principle already mentioned, it is unimportant which of these two we select, the choice not affecting the value of the transversor $\gamma \div \alpha$; since a change in this choice can only invert *both, at once,* of the directions to be finally compared. Assuming then $\beta = +$i, we easily find that we are to assume, at the same time, $\alpha = -$j, and $\gamma = -$k, in order that we may have

$$k \times \alpha = \beta = \mathrm{i},\ j \times \beta = j \times \mathrm{i} = \gamma\,;$$

and thus we find that the required product is

$$j \times k = \gamma \div \alpha = (-\mathrm{k}) \div (-\mathrm{j}) = i.$$

In like manner, to determine the value of $k \times i$, we may assume

$$\beta = +\mathrm{j},\ \ \alpha = -\mathrm{k},\ \ \gamma = -\mathrm{i},$$

and we find that

$$k \times i = (-\mathrm{i}) \div (-\mathrm{k}) = j.$$

73. On the other hand, to find the value of $j \times i$, although we may *still* suppose, as in the example of articles 65, &c., that the versum β is directed vertically upward, we must then *vary* the directions of α and γ from those which were employed in that example; for if we take $\beta = +\mathrm{k}$, we must take $\alpha = +\mathrm{j}$, and $\gamma = +\mathrm{i}$, in order that we may have the relations,

$$i \times \alpha = \beta = +\mathrm{k},\ j \times \beta = j \times (+\mathrm{k}) = \gamma.$$

The telescope is now to be conceived as being originally directed to the west; as being next elevated to the zenith, by a rotation in the plane of the prime vertical, of which the agent or versor is i; and as being finally depressed to the south point of the horizon, by operating with the proversor j. It has, therefore, in this case, been caused upon the whole to *retrograde* (and not to advance) in azimuth through a quadrant, since it has been moved from the west to the south. Or we might assume

$$\alpha = -\mathrm{j},\ \ \beta = -\mathrm{k},\ \ \gamma = -\mathrm{i},$$

because

$$i \times (-\mathrm{j}) = (-\mathrm{k}),\ j \times (-\mathrm{k}) = -\mathrm{i}\,;$$

that is, we might conceive the telescope to be first depressed by the versor i from the east to the nadir, and then elevated by the proversor j from the nadir to the north point; but we should still have, on the whole, a *retrogression* of a quadrant in azimuth, or a *left*-handed motion (from east to north) through a right angle, round an axis directed vertically upwards. Thus,

$$j \times i = (+\mathrm{i}) \div (+\mathrm{j}) = (-\mathrm{i}) \div (-\mathrm{j})\,;$$

but also (by 72 and 60),

$$k \times (-\mathrm{j}) = (+\mathrm{i}), \text{ and } (-) \times (+\mathrm{i}) = (-\mathrm{i});$$

whence it follows that

$$(-\mathrm{i}) = (-) \times k \times (-\mathrm{j}), \quad (-\mathrm{i}) \div (-\mathrm{j}) = (-) \times k,$$

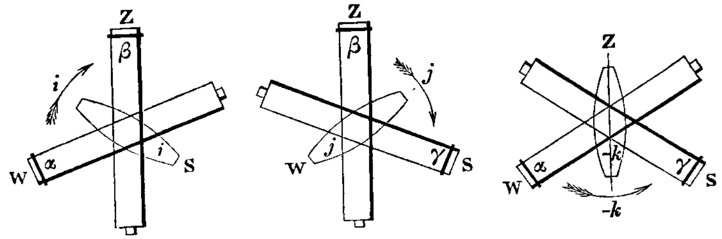
and finally that

$$j \times i = (-) \times k.$$

In words this comes to substituting for the quadrantal retrogression in azimuth a quadrantal *advance*, succeeded by an *inversion* of the telescope.

74. But we may *also* conceive the motion from east to north, or from west to south, to be effected by a *right*-handed rotation through a quadrant, performed round a *downward* axis; and in *this* view, the *transversor* in the present question is seen to be a line in the direction of $-\mathrm{k}$, so that it may conveniently be denoted by the symbol $-k$, as is exhibited in figure 14.

Fig. 14.



We may then write also,

$$j \times i = -k;$$

and in fact this shorter notation is seen to harmonize with the formula recently obtained. It is proper, however, to observe that we have thus been conducted to *one important* DEPARTURE (the only one, indeed, that has hitherto offered itself to our attention) *from the rules or mechanism of common* ALGEBRA. For we have been led to conclude the *two* CONTRASTED *results*:

$$i \times j = k; \; j \times i = -k;$$

which shew that (in the present system) the multiplication of *versors* among themselves is NOT generally a *commutative operation:* or that the ORDER of the *factors* is *not indifferent* to the result. In fact we have been led to *express* thus a THEOREM OF ROTATION, which is indeed very simple, but is, at the same time, very important, and which there is consequently an advantage in having so short a mode of formulizing: namely, the theorem that *two rectangular and quadrantal rotations compound themselves into a third quadrantal rotation, rectangular to both the components, but having one or other of two opposite directions* (or characters, as right-handed or left-handed, round one axis), *according as the composition has been effected in one order or in the other.* It is thus that, for example, in figs. 11, 12, 13, if the rotation denoted by j be *followed* by that denoted by i, the telescope has been seen to be turned upon the whole from north to east, its intermediate position being upward; whereas the same telescope would (as we also saw) be brought *back* from the east to the north, through an intermediate and downward direction, if the rotation i were performed *first*, and *afterwards* the rotation j; or would be brought, as in fig. 14, from a westward to a southward position. It is easy to deduce, on the same plan, the analogous equations,

$$k \times j = -i, \quad i \times k = -j,$$

which are *contrasted* respectively, in the same way, with the equations

$$j \times k = i, \quad k \times i = j;$$

and in which $-i$ is a versor with a northward axis of right-handed rotation, and $-j$ is another versor, with an eastward axis of a rotation likewise right-handed. Or we may write (on the plan of the last article) these other and equivalent formulæ:

$$k \times j = (-) \times i; \quad i \times k = (-) \times j;$$

which would express that the old resultant *rotations* round south and west (in 72) were now to be succeeded by *inversions.*

75. We have not yet considered the *squares* of the symbols i, j, k, or the products of *equal* versors. But we have seen (in 73 and 69), that

$$i \times (+\mathrm{j}) = +\mathrm{k}, \text{ and } i \times (+\mathrm{k}) = -\mathrm{j} = (-1) \times \mathrm{j};$$

by combining which two results it follows that

$$i \times i \times \mathrm{j} = (-1) \times \mathrm{j},$$

or that

$$i \times i = -1.$$

The same conclusion would have followed, if we had twice successively operated by i on the line $-\mathrm{j}$, or on either of the two lines $\pm \mathrm{k}$. In general it is clear that if any line in the prime-vertical (or in any other) plane receive two successive and similar quadrantal rotations, its direction is thereby on the whole inverted or reversed, or multiplied by -1. For the same reason, we have, in like manner, the values:

$$j \times j = -1\,;\ k \times k = -1.$$

We may also write more concisely (compare art. 60),

$$i \times i = j \times j = k \times k = (-)\,;$$

and may say that these three quadrantal versors i, j, k, together with their own opposites, $-i, -j, -k$, are SEMI-INVERSORS, or produce each a *semi-inversion.* Indeed we see more generally that *every other* QUADRANTAL VERSOR with *any* ARBITRARY AXIS *in space*, is, in like manner, a SEMI-INVERSOR, and may be regarded as a *geometrical square root of negative unity;* or even as a square root of *minus*, when "minus" is treated as a *factor:* so that *every such versor* may be considered as *included among the interpretations of the symbol* $\sqrt{-1}$ or $(-)^{\frac{1}{2}}$; at least if we suppose, for the present, each such versor to operate on a line *perpendicular to itself*, or perpendicular to the axis of that quadrantal rotation of which the versor is conceived to be the agent.

76. It may have been noticed that we have not only the six formulæ:

$$\begin{cases} i \times j = k, & j \times k = i, \quad k \times i = j, \\ j \times i = -k, & k \times j = -i, \quad i \times k = -j, \end{cases}$$

considered as results of the *multiplication of versors*, or of the *composition of rotations*, but also the closely analogous formulæ,

$$\begin{cases} i \times \mathrm{j} = \mathrm{k}, & j \times \mathrm{k} = \mathrm{i}, \quad k \times \mathrm{i} = \mathrm{j}, \\ j \times \mathrm{i} = -\mathrm{k}, & k \times \mathrm{j} = -\mathrm{i}, \quad i \times \mathrm{k} = -\mathrm{j}, \end{cases}$$

considered as the six results of so many *single versions*, and *not* of versions *compounded* among themselves. These two sets of results correspond to different conceptions and constructions, and are not to be confounded with each other. We saw, for instance (in connexion with the figures 11, 12, 13), that the formula $i \times j = k$ expressed (as above interpreted) the result of a process, whereby a telescope was first elevated from a northward to a vertical position, and then depressed to an eastward one, being thereby caused upon the whole to advance through a quadrant of azimuth. But the formula $i \times \mathrm{j} = \mathrm{k}$ (which occurred in art. 73, the line j being there denoted by a), expressed, at least according to the interpretation already given, that a telescope originally directed towards the west would be elevated to the zenith, if it were caused to revolve right-handedly through a quadrant round an axis directed to the south (as in the first part of figure 14). The signification of the *one* formula ($i \times j = k$) has thus been made to depend on the consideration of *three* quadrantal rotations, in three rectangular planes; whereas the signification of the *other* formula ($i \times \mathrm{j} = \mathrm{k}$) has been made to depend on the consideration of a *single* rotation of this sort. Yet the two results are by no means *unconnected* geometrically, nor is it *accidental* that their symbolic expressions have so close a resemblance to each other; for this *symbolical analogy* arises from, and embodies, a general *theorem of rotation*. And I conceive that we may *now* legitimately, and with advantage, avail ourselves of the same analogy, or of the theorem to which it corresponds, to *dispense* with that *symbolic distinction* which has been above observed, between the three quadrantal *versors* i, j, k, and the three *lines*, i, j, k, which have respectively the directions of their three axes. Dismissing, therefore, or suspending, the use of the roman letters i, j, k, I propose now to regard the formula $i \times j = k$, as being the *common expression of the two connected results* relative to rotation, of which one was illustrated by the three figures 11, 12, 13, and the other by the first part of figure 14. And in like manner, each of the five other formulæ of the same sort, respecting the binary products of i, j, k, as for example, the formula $j \times k = i$, will come to be regarded as the *common expression* of *two* distinct but connected results; one relative to a certain composition of versions, and the other relative to a single rotation. It is clear that similar remarks apply to the comparison of such results of *division* of rays, and of *decomposition* of versions, as are expressed by the following formulæ:

$$\mathrm{i} = \mathrm{k} \div \mathrm{j}\,; \quad i = k \div j;$$

and by others analogous thereto.

77. In this manner we may be led to regard the three italic letters i, j, k, as symbols of the *same three* LINES which were lately denoted by the three roman letters i, j, k. Or *rather*, for the sake of a somewhat greater *generality*, in future applications, we shall *now* say that i, j, k, *may* be regarded as symbols of ANY THREE MUTUALLY RECTANGULAR AND EQUALLY LONG LINES, whose common length is still supposed to be the UNIT OF LENGTH; while the ROTATION, *round* the first (i), *from* the second (j), *to* the third (k), is POSITIVE; that is (as we shall still suppose) *right-handed:* these last suppositions being a little more general than those of art. 65, in virtue of which the three lines i, j, k, were respectively a southward, a westward, and an upward foot. And, on the other hand, we are conducted to regard each of these three right lines, i, j, k, and similarly EVERY OTHER UNIT LINE in space, as being a QUADRANTAL VERSOR; whose *operation*, on any right line in a plane perpendicular to itself, has the effect of TURNING this latter line THROUGH A RIGHT ANGLE, towards the RIGHT HAND, in the same PERPENDICULAR PLANE.

78. Indeed this VIEW of the directional or GRAPHIC OPERATION of *one* right line on *another* line *perpendicular* thereto, whereby that operation is considered as producing or determining, by a rotation towards a given *hand*, a *third line* perpendicular to *both*, appears to be so *simple* in itself, and so intimately connected with whatever is most CHARACTERISTIC in the whole *conception* of TRIDIMENSIONAL SPACE, that we might have been pardoned if we had chosen to *set out* with it, and to DEFINE that *such* should be regarded, in our system, as the operation of *multiplying one of two rectangular lines by another*, when DIRECTIONS alone were attended to. And then the CONTRAST between the two formulæ,

$$i \times j = k, \quad j \times i = -k,$$

or the *non-commutative character* of this sort of geometrical multiplication, would have offered itself to our notice, even more simply than in art. 74; as expressing, for example, that if a westward line be turned right-handedly through a right angle, round a southward axis, it is ELEVATED to the zenith; but that if (by an *interchange* of operator and operand) a southward line be turned, in like manner, round a westward axis, through a quadrant, and towards the right-hand, *it* is, on the contrary, DEPRESSED to the nadir. And so many other consequences could be drawn from the same simple conception of this *directional operation of line on line*, that it might not be too much to say, that the whole Theory of QUATERNIONS, or that all the symbolical and geometrical properties of quadrinomial expressions of the form $w+ix+jy+kz$, where w, x, y, z are any FOUR SCALAR CONSTITUENTS (four positive or negative numbers), while i, j, k are THREE RECTANGULAR VECTOR UNITS, would admit of being systematically developed from the supposed DEFINITION, above mentioned, of this *case* of the geometrical and *graphic multiplication* of lines; at least if this were combined with those *other* and *earlier* definitions of geometrical addition and subtraction, which other definitions (as was noticed in art. 36) are *not peculiar* to quaternions, but are *common* to *several* systems of application of symbols to geometry. But it has seemed to me that the subject allowed of its being presented to you under a still clearer light, and with a still closer philosophic unity, by the adoption of the plan on which these Lectures have hitherto been framed, and on which it is my purpose to pursue them, if favoured for some time longer with your attention.

LECTURE III.

79. THE two preceding Lectures, Gentlemen, will be found, I think, to have advanced us, in no inconsiderable degree, towards a correct and clear understanding of the *principles* of the Calculus of Quaternions: since they have contained an exposition of the *primary* (and of some of the chief *derivative*) *significations* attached, in that Calculus, to the four elementary *signs* $+ - \times \div$, or to the four fundamental *operations* of Addition, Subtraction, Multiplication, and Division, when viewed in connexion with Geometry. Those primary significations (in the view thus taken of them) have indeed been *stated*, at first, in a very *general* and somewhat *metaphysical* manner; but they have since been *illustrated* by so many and such simple *examples*, geometrical or astronomical, combined with the exhibition, in some cases, of appropriate models and *diagrams*, that the seeming vagueness or obscurity, whatever it may have been, of those early statements (in art. 5), may be hoped to have been, by this time, sufficiently done away. We must, however, now proceed to develope still farther the same principles, and to apply them to new questions, in order to render still more manifest their geometrical meaning and utility. We may not indeed be obliged to enlarge, except in a few instances, the nomenclature or VOCABULARY of the science, which some may think already too copious; but its NOTATION will require to be extended and illustrated by new definitions and examples. The CONCEPTIONS themselves must be still further unfolded and *combined;* and the SYMBOLS by which they are to be embodied and expressed must be shewn to be the elements of a CALCULUS, possessing, on several important points, its own appropriate RULES; although aiming in many other respects, and indeed wherever this can be done without sacrifice of

its peculiar features, to render available, in conjunction with its own new usages, the results and habits of Algebra. More general processes for geometrical Multiplication and Division must be exhibited, than have been given in the foregoing Lecture; and these must be combined with those already stated, for geometrical Addition and Subtraction. And above all, it will be indispensably required by the plan of the present Course, that we should soon proceed to consider more closely than we have hitherto done, the questions, *What* IS, in this System, *a* QUATERNION ? and *On what grounds is it* SO CALLED ?

80. The general notion of *multiplication*, or of FACTION, in geometry, proposed in the foregoing Lecture, has been, that it is an *act* or process which operates 1st, on the *length* of a ray; or 2nd, on its *direction;* or 3rd, on *both* length and direction at once. The *multiplier* or FACTOR has been conceived to be the *agent* or the *operator* in this act or process; and the multiplication of any two factors *among themselves*, in any assigned *order*, has been conceived to correspond to the *composition* of two *successive* acts of faction, and to the *determination* of the *agent* in the resulting act of *transfaction*. And the *mark* or characteristic of such faction, or of such composition of factions, has been with us the familiar sign $\times$, pronounced or read, as usual, by the word INTO. As *examples* of such FACTORS in geometry, we have as *yet* considered only the *four* following classes: I. TENSORS or signless numbers, such as 2, 3, 6, $\frac{1}{2}$, $\sqrt{2}$, which operate only *metrically* on the lengths of the lines which they multiply, and which are to be combined among themselves, as factors, by arithmetical multiplication, or by the laws of the composition of ratios; II. SIGNS, namely (+) and (−), regarded as marks of nonversion and inversion, which operate (as such) only to preserve or to reverse the direction of a line, and are combined among themselves according to the usual rule of the signs; III. SCALARS, or sign-bearing numbers, such as − 2 or + 6, which are simply the *reals* of ordinary algebra, and are combined with each other as factors according to the known rules of algebraic multiplication, while each may be regarded as being *itself* the product of a tensor and a sign, and may at once alter the length of a line in a given ratio, and *also* nonvert or invert its direction; IV. VECTOR-UNITS,