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of a vector is a *right angle*; the angle of a *positive* scalar is *zero*, and the angle of a *negative* scalar is *two right angles*; with the single *exception* of *powers of negatives* (for which *powers*, as well as for their *bases*, the *axes are indeterminate*), the same definition assigns a *determinate quaternion* as the value of the t^{th} power of any proposed quaternion q; and the equation $q^u q^t = q^{u+t}$ is satisfied, each member representing a quaternion, of which the versor has the effect of turning a line perpendicular to the axis of q through an amount of rotation represented by $(u+t) \angle q$, ...

§ XXXIII. On the other hand, although the ROTATION produced by the operation of the *power* q^t is now correctly and *definitely* expressed by the *product* $t \times \angle q$, yet because this product is not generally confined between the *limits* 0 and π, we cannot now consider it as being *generally* equal to the *angle of the power*, because we have agreed (in § XXXII.) to *confine* the ANGLE of *every quaternion*, and therefore of the power q^t among the rest, *within* those limits; thus with the present DEFINITE SIGNIFICATION of the mark $\angle$, we must *not* write generally $\angle (q^t) = t \times \angle q$, but rather $\angle (q^t) = 2n\pi \pm t \angle q$, the *axis* of the *power* being in the *same* direction as the axis Ax . q of the *base*, or else in the *opposite* direction, according as it becomes necessary to take the upper or the lower sign; the *square root*, $q^{\frac{1}{2}}$, of a (non-scalar) quaternion is *acute-angled*, and so are the *cube-root*, $q^{\frac{1}{3}}$, &c., while the *axes* of these *roots* coincide with the axis of their common power; but the *square* q^2 of an *obtuse*-angled quaternion q has its angle $\angle (q^2)$ equal to the *double of the supplement* of the obtuse angle $\angle q$, and has its axis in the direction *opposite* to that of the axis Ax . q; with this *definite* view of powers and roots, although *three* distinct quaternions may have *one common cube*, yet *only one* of them is (by eminence) *the cube-root* of that cube; examples: in like manner the symbol $(q^2)^{\frac{1}{2}}$ denotes now definitely $+q$, or $-q$, according as the angle of q is acute or obtuse; $(\rho^2)^{\frac{1}{2}}$ denotes a vector, with a length $= \mathrm{T}\rho$, but with an *indeterminate direction*, because ρ^2 is a negative scalar; we must *not* now write generally $(q^t)^u = q^{ut}$, but *may* establish this modified formula, $(q^t)^u = (\mathrm{Ax} . q)^{4nu} . q^{ut}$, ...

§ XXXIV. *Reciprocals* and *conjugates* of quaternions (compare §§ XXIV., XXX.):

$$\mathrm{T}(q^{-1}) = (\mathrm{T}q)^{-1} = \mathrm{T}q^{-1},\ \mathrm{U}(q^{-1}) = (\mathrm{U}q)^{-1} = \mathrm{U}q^{-1};$$
$$\angle (q^{-1}) = \angle q,\ \mathrm{Ax} . (q^{-1}) = -\mathrm{Ax} . q;\ \mathrm{U}q^{-1} = \mathrm{KU}q = \text{reversor};$$
$$\angle \mathrm{KU}q = \angle \mathrm{U}q,\ \mathrm{Ax} . \mathrm{KU}q = -\mathrm{Ax} . \mathrm{U}q;$$
$$\angle \mathrm{K}q = \angle q,\ \mathrm{Ax} . \mathrm{K}q = -\mathrm{Ax} . q,\ \mathrm{TK}q = \mathrm{T}q;$$

the reciprocal and conjugate of q may be thus expressed,

$$q^{-1} = \mathrm{T}q^{-1} . \mathrm{KU}q,\ \mathrm{K}q = \mathrm{T}q . \mathrm{U}q^{-1};$$

in general $q\mathrm{K}q = \mathrm{T}q^2$, so that the *product of any two conjugate quaternions is a positive scalar*, namely, the *square of their common tensor*; $\mathrm{T}q = (q\mathrm{K}q)^{\frac{1}{2}}$, $\mathrm{U}q = \pm (q \div \mathrm{K}q)^{\frac{1}{2}}$, according as $\angle q \lesseqgtr \frac{\pi}{2}$; exam-

LECTURE V.

(Articles 175 to 250 ; Pages 186 to 240.)

ASSOCIATIVE PRINCIPLE FOR THE MULTIPLICATION OF THREE LINES IN SPACE; QUATERNION VALUES OF THEIR TERNARY PRODUCTS, $\beta\alpha\gamma$, AND FOURTH PROPORTIONALS, $\beta\alpha^{-1}\gamma$; VALUES OF ijk, kji; GENERAL CONSTRUCTION FOR THE PRODUCT OF TWO VERSORS, BY A TRANSVECTOR ARC UPON A SPHERE.

LECTURE VI.

§ LV. One form of the *equation of the tangent plane* at A to the sphere ABCD is the following:

$$AB . BC . CD . DA . AP = AP . DA . CD . BC . AB ;$$

the two equations,

$$AB . BC . CD . DE . EA = EA . DE . CD . BC . AB,$$

and

$$AB . BC . CD . DA . AE = AE . DA . CD . BC . AB,$$

must therefore be incompatible, except under the supposition that either the point E coincides with A, or that the four points A, B, C, D are coplanar; in fact when the distributive principle shall have been established (in § LXXV.), it will become clear that the addition of these two equations gives

$$AB . BC . CD \times AE . EA = AE . EA \times CD . BC . AB,$$

and therefore that either

$$AE^2 = 0, \quad AE = 0, \quad E = A,$$

or else

$$AB . BC . CD = CD . BC . AB,$$

which are respectively (compare § XXXVIII.) conditions of coincidence and coplanarity; problem of inscription in a given sphere, of a gauche quadrilateral ABCD, whose four successive sides AB, ... DA shall be respectively parallel to four given radii OI, OK, OL, OM; problem of expressing an n^{th} radius, OP_n, or ρ_n, of a given sphere, considered as a function of an initial radius OP or ρ, and of n other radii, $OI_1, \ldots OI_n$, or $\iota_1, \ldots \iota_n$, to which the n successive and rectilinear *chords* $PP_1, \ldots P_{n-1} P_n$ are required to be parallel; if α and β be any two equally long and diverging lines, OA, OB, and if γ have either of the two opposite directions of the lines AB, BA connecting their extremities, then $\beta = -\gamma\alpha\gamma^{-1}$; hence in the recent question, $\rho_1 = -\iota_1\rho\iota_1^{-1}$, $\rho_2 = -\iota_2\rho_1\iota_2^{-1}$, &c., and if we introduce the quaternion, $q_n = \iota_n \ldots \iota_2\iota_1$, the solution of the problem will be expressed by the formula $\rho_n = (-)^n q_n \rho q_n^{-1}$; the same expression will hold good, if we regard the quaternion q_n as the continued product

$$q_n = (\alpha_n - \rho_{n-1})\,(\alpha_{n-1} - \rho_{n-2}) \ldots (\alpha_1 - \rho),$$

of the *n first segments* PA_1, P_1A_2, ... &c., of the *n successive chords*, on which A_1, A_2, &c., are n points arbitrarily taken, but not supposed to be situated upon the surface of the sphere; relation to a conical rotation (see § L.); EQUATION OF CLOSURE, $\rho_n = \rho$; for an inscribed and *even-sided polygon*, $\rho q_n = q_n \rho$, $\mathrm{Ax} . q_n \parallel \rho$, with inclusion of the limiting case for which the product q_n is a scalar; for an *odd-sided* polygon, $\rho q_n = -q_n\rho$, and the same product q_n must reduce itself to a vector $\perp \rho$; these last results agree with those of § LIV.; if, in a sphere, the five successive *sides* of an *inscribed gauche pentagon*, ABCDE, be respectively *parallel* to the five *radii* drawn to the five *corners* of a *superscribed spherical pentagon*, IKLMN, then the *fifth corner* N of the *second* pentagon is situated somewhere *upon that great circle* FH, of which a portion coincides with the

LECTURE VII.

ADDITION AND SUBTRACTION OF QUATERNIONS; SEPARATION OF THE SCALAR AND VECTOR PARTS; NOTATIONS S AND V; DISTRIBUTIVE PRINCIPLE OF MULTIPLICATION OF QUATERNIONS; NEW PROOF OF THE ASSOCIATIVE PRINCIPLE; GEOMETRICAL APPLICATIONS OF THESE PRINCIPLES, INCLUDING SOME NEW GENERATIONS AND PROPERTIES OF THE ELLIPSOID; NEW REPRESENTATIONS OF LOCI; CONNEXIONS OF QUATERNIONS WITH CO-ORDINATES, DETERMINANTS, TRIGONOMETRY, LOGARITHMS, SERIES, LINEAR AND QUADRATIC EQUATIONS, DIFFERENTIALS, AND CONTINUED FRACTIONS; INTRODUCTION OF THE BIQUATERNION.

which *quotient* is in general (see §§ VI., XX.) a QUATERNION, as the alue of the proposed SUM,

$$1+k=(1i+ki)\div i=(i+j)\div i;$$

the *effect* of $1+k$, as a *factor*, is to change the *side* of a horizontal square to that *diagonal* of the same square which is more advanced than it in azimuth by 45°;

$$\mathrm{T}\,(1+k)=2^{\frac{1}{2}},\ \ \mathrm{U}\,(1+k)=k^{\frac{1}{2}},\ \ 1+k=2^{\frac{1}{2}}\,k^{\frac{1}{2}};$$

this plan of *interpretation* of the symbol $1+k$ is analogous to that employed in the calculus of finite differences for the interpretation of the symbol $1+\Delta$, in which *also* the two summands appear at first as *heterogeneous*, but are *incorporated* by being made to operate on one *common function* fx; more elementary illustration of the process; in general the symbol $w+\rho$, where w denotes a scalar, and ρ a vector, can on the same plan be interpreted as a *quotient of two lines*, and therefore as a *quaternion*, by taking some line $\alpha \perp \rho$, and *defining* that $w+\rho=(w\alpha+\rho\alpha)\div\alpha$, when $w\alpha$ and $\rho\alpha$ are *lines;* addition of this sort is a perfectly *definite* operation, and has the *commutative* character, $w+\rho=\rho+w$, ...

§ LXIII. Conversely, an *arbitrary quaternion* q can always be *definitely decomposed* into *two parts*, such as w and ρ, of which one shall be a *number* and the other a *line*, although it is possible that one of these parts may vanish; if $q=\beta\div\alpha$, and if we *decompose the dividend line* β by *projection* into two *partial vectors*, or summand lines, β', β'', respectively *parallel* and *perpendicular* to the divisor line α, and divide *each part* separately by that line α, the partial quotients thus obtained will be respectively *the scalar part* and *the vector part* of the *total quotient* or quaternion q; introducing then the letters S and V, as *characteristic of the two operations* of TAKING THE SCALAR and TAKING THE VECTOR of a quaternion, we shall have $\mathrm{S}\,(w+\rho)=w$, $\mathrm{V}\,(w+\rho)=\rho$, and $\mathrm{S}\,(\beta\div\alpha)=\beta'\div\alpha$, $\mathrm{V}\,(\beta\div\alpha)=\beta''\div\alpha$, if $\beta=\beta'+\beta''$, $\beta'\parallel\rho$, $\beta''\perp\rho$; $q=\mathrm{S}q+\mathrm{V}q=\mathrm{V}q+\mathrm{S}q$, $1=\mathrm{S}+\mathrm{V}=\mathrm{V}+\mathrm{S}$; also (compare § XVI.), $\mathrm{S}^2=\mathrm{S}$, $\mathrm{SV}=\mathrm{VS}=0$, $\mathrm{V}^2=\mathrm{V}$; thus, $\mathrm{S}w=w$, $\mathrm{S}\rho=0$, $\mathrm{V}w=0$, $\mathrm{V}\rho=\rho$; *conjugate quaternions* have *equal scalars* but *opposite vectors*, $\mathrm{SK}q=+\mathrm{S}q$, $\mathrm{VK}q=-\mathrm{V}q$, $\mathrm{SK}=\mathrm{S}$, $\mathrm{VK}=-\mathrm{V}$; $\mathrm{K}\,(w+\rho)=w-\rho$ (§ XXIII.); $\mathrm{K}q=\mathrm{S}q-\mathrm{V}q$, $\mathrm{K}=\mathrm{S}-\mathrm{V}$; $\mathrm{TK}=\mathrm{T}$ (§ XXXIV.), $\mathrm{T}\,(w+\rho)=\mathrm{T}\,(w-\rho)=(w^2-\rho^2)^{\frac{1}{2}}$ (§ XXII.); if x be a scalar, $\mathrm{V}x=0$, then $\mathrm{S}\,.\,xq=x\mathrm{S}q$, $\mathrm{V}.\,xq=x\mathrm{V}q$; for example,

$$\mathrm{S}\,(-q)=-\mathrm{S}q,\ \ \mathrm{V}\,(-q)=-\mathrm{V}q;$$
$$\mathrm{S}\,(-\mathrm{K}q)=-\mathrm{S}q,\ \ \mathrm{V}\,(-\mathrm{K}q)=+\mathrm{V}q,\ \ -\mathrm{K}=\mathrm{V}-\mathrm{S};$$
$$x\,(w+\rho)=xw+x\rho;\ \ \mathrm{ST}q=+\,\mathrm{T}q,\ \ \mathrm{VT}q=0;$$
$$\mathrm{S}q=\mathrm{T}q\,.\,\mathrm{SU}q,\ \ \mathrm{V}q=\mathrm{T}q\,.\,\mathrm{VU}q;\ \ \mathrm{VU}q=\mathrm{UV}q\,.\,\mathrm{TVU}q;$$
$$\mathrm{UV}q=\mathrm{Ax}\,.\,q,\ \ (\mathrm{UV}q)^2=-1,\ \ \mathrm{UV}q=\sqrt{-1};$$

quaternions are connected with trigonometry, by the relations,

$$\mathrm{SU}q=\cos\angle q,\ \ \mathrm{TVU}q=\sin\angle q;$$

equation be combined with the equation of a *new plane*, $S . \rho\gamma^{-1} = 1$, the resulting system represents a PLANE CONIC, considered as a *curve in space;* the equation of the cone may also be thus written,

$$S \frac{\rho}{\beta^{-1}} \; S \frac{\alpha^{-1}}{\rho} = 1 ;$$

under this form it gives the SUBCONTRARY CIRCULAR SECTION of the cone, namely, as the intersection of the sphere described on α^{-1} as diameter, with the plane $S . \rho\beta = 1$; the *parallel plane through the vertex,* $S . \rho\beta = 0$, *touches the former sphere* $S . \beta\rho^{-1} = 1$, which contained the *former circular base;* this latter plane, and the plane $S . \rho\alpha = 0$, are the TWO CYCLIC PLANES of the cone; the equations of these two planes may also be thus written, $S . \beta\rho = 0$, $S . \alpha\rho = 0$; for in general (by §§ XV., LXIII.), $S . \rho\alpha = SK . \rho\alpha = S . \alpha\rho$; thus, in taking the *scalar of the product* of *any two vectors,* we are allowed to *alter their order;* more generally it will be found (see § LXXXIX.), that *under the sign* S *we may alter* CYCLICALLY *the* ORDER *of any* NUMBER *of factors,* even if those factors be *quaternions;* a SPHERICAL CONIC may be expressed by combining either of the two forms above assigned for the equation of the cone with any one of the three following forms for the equation of the CONCENTRIC SPHERE,

$$T\rho = c, \quad \rho^2 + c^2 = 0, \quad S \frac{\rho - \gamma}{\rho + \gamma} = 0 ;$$

γ is here the vector of some one point upon the sphere, and c is the length of the radius; we might also represent the same concentric sphere by the equation $T\rho = T\gamma$, or $\rho^2 = \gamma^2$; one CYCLIC ARC may be represented by the two equations $S . \alpha\rho = 0$, $T\rho = c$, and the *other* cyclic arc by the equations, $S . \beta\rho = 0$, $T\rho = c$, ... Articles 416 to 421; Pages 402 to 407.

§ LXVI. If a given sphere with a for radius have its centre at the origin O, and if we conceive T to be a sought point of contact of the sphere with a rectilinear tangent from a given external point S, and make $\sigma = OS$, $\tau = OT$, we shall have the two equations $\tau^2 = -a^2$, $S . \sigma\tau^{-1} = 1$, the first denoting the *given* sphere round O, and the second an *auxiliary* sphere on OS; the POLAR PLANE of the point S, or the plane of which S is the POLE, with respect to the given sphere, is the plane of the circle of intersection of the two spheres, and its equation (obtained by suitably multiplying *their* equations) is $S . \sigma\tau = -a^2$, or $S . \tau\mu^{-1} = 1$, if we make $\mu = OM = -a^2\sigma^{-1}$; τ is here treated as a variable vector, but σ and μ as fixed vectors; $U\mu = U\sigma$, $T\mu = a^2 T\sigma^{-1}$; M is the *centre of the circle of contact* of the given sphere with the ENVELOPING CONE of tangents drawn from S; if $\rho = OP$ be the variable vector of a point P upon this cone, then

$$\{(S . \sigma(\rho - \sigma)\}^2 = (\sigma^2 + a^2)\,(\rho - \sigma)^2 ;$$

but a simpler form of the *equation of the enveloping cone* will be assigned afterwards (in § LXXVII.); the cone which cuts this enveloping cone perpendicularly along the above-mentioned circle of contact, and has its ver-

as the *angle of the versor*, $TVq' = \angle Ur$, with that *definite sense* of the word "angle," which was proposed in § XXXII.; if the versor Ur have been given, or found, under the form, FVq, and if $TVq > \pi$, whereas $TVq' \not> \pi$, it is proposed to consider Vq', and *not* Vq, as the (principal) *value of the* INVERSE EXPONENTIAL FUNCTION, or to write $F^{-1} Ur = Vq'$; with this *definite signification* of that function we may therefore write, $\angle r = \angle Ur = TF^{-1} Ur$; also $UF^{-1} Ur = UVr = Ax . r$, and $F^{-1} Ur = UVr . \angle r$; we may also definitely interpret $F^{-1} Tr$ as $= lTr =$ that positive or negative number, or zero, which is the natural or Napierian *logarithm* of Tr; and more generally we may agree to call the *inverse exponential function* (or the IMPONENTIAL) $F^{-1} r$, OF ANY QUATERNION r, the LOGARITHM of that quaternion, and to *interpret* it *definitely* as follows:

$$lr = F^{-1} r = F^{-1} Tr + F^{-1} Ur = lTr + UVr . \angle r;$$

the *scalar of the logarithm* of a quaternion is thus the *logarithm* of the *tensor*, and the *vector of the logarithm* is the *logarithm of the versor;* in symbols,

$$Slr = lTr,\ Vlr = lUr = UVr . \angle r$$

$=$*product of axis and angle;* that is, *the vector of the logarithm of any quaternion* is *constructed*, in our system, by the REPRESENTATIVE ARC RECTIFIED, and placed PERPENDICULARLY TO THE PLANE, or *in the* DIRECTION OF THE AXIS, of the quaternion; the *logarithm of a given quaternion*, thus interpreted, is *generally a* DETERMINED *quaternion*, but becomes *partially indeterminate*, when the given quaternion *degenerates* to a *negative number*, or to *zero;* we may agree to employ the usual symbol e^q, as a *concise expression* suggested by algebra (compare § XCIII.), for the *series* $1 + q + \frac{1}{2} q^2 +$ &c., or for the *direct* exponential function Fq; a POWER of a quaternion, with a QUATERNION EXPONENT, may then in general be *definitely interpreted* by means of the formula,

$$q^r = F(rF^{-1} q) = e^{rlq};\ \text{examples},\ j^i = k,\ j^j = e^{-\frac{\pi}{2}};$$

expressions for the tensor and versor of the *general power*, q^r; MENSOR of a quaternion, $Mq = lTq$ (this notation and nomenclature are not insisted on); *definite interpretation* of the *logarithm of a given quaternion to a given* QUATERNION BASE, namely, as the *quotient of their two natural logarithms;* $\log_q . q' = lq' \div lq$; this GENERAL LOGARITHM *might be so interpreted* as to involve *two arbitrary integers*, as in some known theories; but we prefer, in this calculus, to *exclude such indetermination by definition*, in this as in other cases, wherever such exclusion is possible; interpretations of the *sine*, *cosine*, and *tangent*, of a quaternion; if we take *two arbitrary quaternions*, q and r, we shall *still* have, as in algebra,

$$e^r e^q = 1 + (r + q) + \tfrac{1}{2}(r^2 + 2rq + q^2) + \&c.;$$

but $r^2 + 2rq + q^2$, &c. will *not* in this calculus be equal to the *square*, &c., of $r + q$, *unless* $rq = qr$, or $V.VrVq = 0$, which will *not generally* happen; *when this condition* of *commutativeness*, of q and r as factors, is *not satisfied*, then if x be any scalar coefficient, supposed to vanish after the per-

q and dq are here *any two quaternions*, Tdq being *not* necessarily *small*, but the positive whole number n being conceived to increase without limit; the *third* quaternion dfq, which results as the limit of this process, is a *function of the two assumed quaternions*, q and dq, of which the particular *form* depends on the form of the *proposed function*, f, but which is always *linear*, or *distributive*, with respect to the quaternion dq; but this differential dfq is *not* in general reducible in *this* calculus, to a product of the form $f'q$. dq, if $f'q$ denote a function of the quaternion q alone; when the function $f(q + \mathrm{d}q)$ can be developed in a *series*, involving *terms* or parts of successively higher and *higher dimensions*, with respect to the quaternion dq, the *part* of this developement which is of the *first* dimension, relatively to dq, is (as in the ordinary differential calculus) the required differential dfq; but it is proposed to *avoid*, in this calculus, adopting *this* as the *fundamental* property of a differential, because the recent *definition* can often be applied more easily than the *developement* can be found; examples; $\mathrm{d} \,.\, q^2 = q \,.\, \mathrm{d}q + \mathrm{d}q \,.\, q$, or more concisely, $\mathrm{d} \,.\, q^2 = q\mathrm{d}q + \mathrm{d}qq$, d$q$ being treated as a *simple symbol*, or as if it were a *single letter;* $\mathrm{d} \,.\, q^{-1} = -q^{-1}\mathrm{d}qq^{-1}$; in differentiating any *product* of quaternions, we simply differentiate each factor *in its own place ; we may extend Taylor's series to quaternions*, under the form $f(q + \mathrm{d}q) = e^{\mathrm{d}}fq$, where d$q$ is treated as constant; examples; . . . . Articles 568 to 573; Pages 569 to 572.

§ XCIX. Geometrical applications; if a vector ρ be a given function ϕt of a variable scalar t, we may express its differential under the *usual* form, $\mathrm{d}\rho = \mathrm{d}\phi t = \phi' t \,.\, \mathrm{d}t = \rho'\mathrm{d}t$, where $\rho' = \phi' t =$ a certain *derived vector*, which is *parallel to the tangent* to the *curve in space*, which is the *locus* of the extremity of ρ; the *length* of this new vector is *unity*, $\mathrm{T}\phi' t = 1$, if the *arc* be the independent variable; in mechanics, if t denote the *time*, and if a second differentiation have given $\mathrm{d}\rho' = \mathrm{d}\phi' t = \phi'' t \,.\, \mathrm{d}t = \rho''\mathrm{d}t$, then ρ' may be called the *vector of velocity*, and ρ'' the *vector of acceleration*, while ρ may be named the *vector of position ;* in geometry, if t be again the arc of the curve, $\rho - \rho''^{-1}$ is the *vector of the centre of the osculating circle*, and ρ'' may therefore be called the *vector of curvature ;* when a *surface* is expressed, as in § LXXXIX., by an equation of the form $f\rho =$ const., where f denotes a *scalar* function, we may then, by cyclical permutation under the sign S (see the same section LXXXIX.), express the *differentiated equation* of that surface under the form $\mathrm{d}f\rho = 2\mathrm{S} \,.\, \nu\mathrm{d}\rho = 0$; the *logic* of this process will be more closely considered in § CI.; ν is a NORMAL VECTOR, and if we oblige it to satisfy the condition $\mathrm{S} \,.\, \nu\rho = 1$, then (compare § LXXXI.) its *reciprocal* ν^{-1} will represent, in length and in direction, the *perpendicular* let fall from the origin of vectors on the *tangent plane* to the surface, so that ν itself may be called, under the same conditions, the *vector of proximity ;* without obliging ν to satisfy the equation $\mathrm{S} \,.\, \nu\rho = 1$, if we only choose it so as to give generally $\mathrm{S} \,.\, \nu\mathrm{d}\rho = 0$, it will still be, as before, a *normal vector*, and this symbol ν may be used to form EQUATIONS OF CLASSES OF SURFACES; thus an *arbitrary cone* (with vertex at origin) may be denoted

$$+i\left(\frac{dv}{dy}-\frac{du}{dz}\right)+j\left(\frac{dt}{dz}-\frac{dv}{dx}\right)+k\left(\frac{du}{dx}-\frac{dt}{dy}\right),$$

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=-\triangleleft^2 v,$$

§ CIX. A DEFINITE INTEGRAL in quaternions may be interpreted as a *limit of a sum;* but, *even* when the function to be integrated remains *finite* between the limits of integration, *still* if the differential factor dq under the sign of integration be *itself* essentially a *quaternion*, then a certain degree of *indetermination* of value of the quaternion integral $\int_{q_0}^{q_1}F(q,\mathrm{d}q)$ arises from the possibility of assuming indefinitely many different *laws of dependence* of the variable quaternion q on a *scalar* variable t, which latter may be supposed to change from 0 to 1, while q changes from one given quaternion value q_0 to another q_1; in this way arises a *new sort of variation of a definite integral*, depending on the *non-commutative* character of multiplication, which may be symbolized by the formula,

$$\delta Q=\delta\int_{q_0}^{q_1}F(q,\mathrm{d}q)=\int_{q_0}^{q_1}\{\delta_q F(q,\mathrm{d}q)-\mathrm{d}_q F(q,\delta q)\};$$

for example,

$$\delta\int fq\mathrm{d}q=\int(\delta fq\,.\,\mathrm{d}q-\mathrm{d}fq\,.\,\delta q),\ \text{if } \delta q_0=0,\ \delta q_1=0;$$

$$u_x = \left(\frac{b}{a+}\right)^x c,$$

where a, b, and c $(= u_0)$ are any three given quaternions, and x is a positive whole number; making

$$v_x = (u_x + q_2)(u_x + q_1)^{-1},$$

we have $v_x = q_2{}^x v_0 q_1{}^{-x}$, where q_1, q_2 are *any two roots* of the quadratic equation $q^2 = qa + b$; examples,

$$\left(\frac{j}{i+}\right)^x 0, \ \left(\frac{j}{i+}\right)^x c, \ \left(\frac{10j}{5i+}\right)^x c, \ \left(\frac{\beta}{a+}\right)^x \rho_0;$$

in the two first of these four examples, the continued fraction has generally a *period of six values*, which may be found at pleasure by employing the two *real quaternion roots* of the quadratic equation $q^2 = qi + j$, namely,

$$q_1 = \tfrac{1}{2}(1 + i + j - k), \ q_2 = \tfrac{1}{2}(-1 + i - j - k);$$

or two *conjugate imaginary solutions* of that quadratic, such as the pair $q_1 = zi - k$, $q_2 = z^{-1}i - k$, where $z = (\cos + \sqrt{-1}\sin)\frac{\pi}{3}$, $\sqrt{-1}$ being the *old* imaginary symbol (compare § CXI.); or the *other pair* of imaginary roots of the same quadratic equation, included in the expression,

$$q = \tfrac{1}{2}(i + k) \pm \tfrac{1}{2}(1 - j)\sqrt{-3};$$

or by *any other selection* of two roots, for instance, by *combining one real and one imaginary root;* the six real quaternion terms of the period, found by any of these combinations of roots, agree with those obtained by actually performing the *divisions* prescribed by the form of the continued fraction; in the third example above cited, of such a fraction, the value does *not circulate*, but (generally) *converges to a limit*, so that

$$\left(\frac{10j}{5i+}\right)^\infty c = 2k - i, \text{ unless } c = 2k - 4i;$$

in this last case, and also in the case when $c = 2k - i$, that is, when c is a real root of the quadratic $c^2 + 5ci = 10j$, the value of the fraction is *constant; geometrical interpretations,* for the case where $c = ix_0 + kz_0$, x_0 and z_0 being regarded as the coordinates of an assumed point P_0 in the plane of ik (or xz); *successive derivation* of other points P_1, P_2, &c., according to a *law* assigned; if the assumed point be placed at *either of two fixed points* F_1, F_2, in the same plane of ik, its position will *not be changed* by this mode of successive derivation; but if P_0 be taken *anywhere else* in the plane, the derivative points will indefinitely *tend to the fixed position* F_2, so that we may write

$$P_\infty F_2 = 0, \ P_\infty = F_2, \text{ unless } P_0 = F_1;$$

law of this approach; continual *bisection of the quotient,* $PF_2 \div PF_1$, of the distances of the variable point P from the two fixed points; theorem of the *two circular segments*, on the *common base* F_1F_2, and containing the

[* The foregoing Analysis of the work into *Sections* did not occur to the author until it was too late to be incorporated with the text: but it has been printed here, as seeming likely to be useful.]

REFERENCES TO THE FIGURES.