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ADDITIONS AND SUBTRACTIONS OF LINES AND POINTS.

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LECTURE II.

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tations of a line : to know fully *what particular act of version* has been performed, we must know through *what angle*, in *what plane*, and towards *which hand* (or round *what axis*, and through *what amount of right-handed rotation*), the line has been made to *turn*,

Articles 65, 66 ; Pages 58 to 61.

§ x. Illustrations from meridional and extra-meridional transit telescopes, and from the theodolite, or other instrument moveable in azimuth ; *non-commutative character of the composition of versions in rectangular planes ;*

$$\begin{aligned} i \times j &= k, & j \times k &= i, & k \times i &= j ; \\ j \times i &= -k, & k \times j &= -i, & i \times k &= -j ; \\ i \times i &= j \times j = k \times k = -1 = (-) ; \end{aligned}$$

every QUADRANTAL VERSOR is a SEMI-INVESOR, and as such is a geometrical square root of negative unity, or of the sign minus ; every such versor is represented, in the geometrical applications of this calculus, by a VECTOR-UNIT, drawn in the direction of the axis of the version : thus the symbols i, j, k come to denote here three rectangular vector-units (supposed usually, in these Lectures, to be in the directions of south, west, and up) ; and the formula $i \times j = k$ is found to receive two distinct but closely connected interpretations, Articles 67 to 78 ; Pages 61 to 73.

LECTURE III.

(Articles 79 to 120 ; Pages 74 to 129.)

OTHER CASES OF MULTIPLICATION AND DIVISION IN GEOMETRY ; CONCEPTION OF THE QUATERNION ; NOTATIONS, K, T, U.

§ XI. Recapitulation ; additional illustrations of the effects of i, j, k , as operators ; multiplication of *any one line in space*, by another *perpendicular* thereto ; the *product* is (in this system) a *third line*, perpendicular to *both* the factors ; its *length* is numerically the product of their lengths ; and the *direction* of the same product-line is obtained from that of the multiplicand line, by a *positive and quadrantal rotation*, performed round the multiplier line as an axis ; *non-commutative character of such multiplication, equation of perpendicularity, $a\beta = -\beta a$, if $\beta \perp a$* ; these results are extensions of those expressed by the formulæ, $ij = k, ji = -k,$

Articles 79 to 82 ; Pages 74 to 79.

§ XII. The *product of a scalar and a vector*, or of a number and a line, is a *line*, of which the length and the direction are very easily assigned, and are found to be independent of the order of the factors ; $aa = aa$; for example, the symbols ix, jy, kz , denote the same three rectangular lines as xi, yj, zk ; namely, when this system is brought into connexion with the Cartesian method of co-ordinates, the three rectangular projections of the line drawn from the origin $(0, 0, 0)$, to the point (x, y, z) ,

Article 83 ; Pages 79, 80.

- § XIII. The *product of two parallel lines* is a *number*, namely, the numerical product of the lengths of the factors; but this number is taken *negatively* or *positively* (in THIS calculus), according as they *agree* or *differ* in their directions; thus, the *SQUARE of EVERY VECTOR* is a *NEGATIVE SCALAR*, $a^2 < 0$ (as we had $i^2 = j^2 = k^2 = -1$); this remarkable result is a simple geometrical consequence of the composition of *two successive and quadrantal rotations about any common axis in space*; *commutative* character of the multiplication of *parallel vectors*, *equation of parallelism*, $a\beta = \beta a$, if $\beta \parallel a$, Articles 84, 85; Pages 80 to 82.
- § XIV. *Powers of unit-vectors*; symbols ι^t, ι^k , where ι is such an unit-line in space, and κ a vector $\perp \iota$; the first of these two symbols (ι^t) denotes a *versor*, not generally quadrantal; the second (ι^k) denotes a *line*, which is formed from κ by a positive and plane rotation of t quadrants, round ι regarded as an axis; examples, Article 86; Pages 82, 83.
- § XV. Multiplication of two *inclined lines*; their product $\kappa\lambda$ (which is afterwards shewn to be a *quaternion*) may also be considered as the *product of a tensor and a versor*; whereof the tensor is the numerical *product of the lengths* of the two factor lines; while the versor has its *axis* in the direction of the axis of *positive* (namely, in these Lectures, *right-handed*) rotation, *from* the multiplier line κ to the multiplicand line λ , and has its *angle* equal to the *supplement* of the angle of this last rotation; examples; versor and re-*versor*; *CONJUGATE VERSORS*, *conjugate products*, *CHARACTERISTIC OF CONJUGATION K*; $K . \iota^t = \iota^{-t}$, $K . \kappa\lambda = \lambda\kappa$, Articles 87 to 89; Pages 83 to 87.
- § XVI. *Resolution of every act of faction* into a *metric* and a *graphic* element, or into an *act of tension*, and an *act of version*; the letters T and U are employed in this calculus as *characteristics of the two separate operations*, of *TAKING THE TENSOR*, and *TAKING THE VERSOR*, or of taking separately the *two factor-elements*, Tq and Uq , of any proposed factor q , or of any product or quotient of two lines, when regarded *as such a factor*; identities, $q = Tq \times Uq = Uq \times Tq$; $T . Uq = 1$, $U . Tq = +$; $T . Tq = Tq$, $U . Uq = Uq$, Article 90; Pages 87 to 89.
- § XVII. The tensor Tq (by §§ VIII., XVI.) is always to be conceived as a *single number*, expressing the *ratio* in which the *factor q changes the length* of the line a on which it operates; but (by §§ IX., XVI.) the *versor* Uq , which may generally be put (see § XIV.) under the form of a *power* ι^t of an *unit-vector* ι , with a *scalar exponent*, t , requires for its complete numerical determination a *system of three numbers*; namely, the number (t) of quadrants contained in the *angle* of the version; and some *two angular co-ordinates* or other equivalent system of two numbers, to fix the *direction in space of the axis* (ι), or to identify on a globe or chart the *star*, or to fix the *region* of infinite space, towards which that axis is pointed; it follows therefore that the lately considered *product of tensor and versor*, $Tq . Uq$, or (see § XVI.) the equivalent *factor q*, *depends upon*, and *conversely includes within itself*, a *SYSTEM OF FOUR NUMBERS*, as necessary

for its complete identification, or full numerical determination ; and therefore that a GEOMETRICAL FACTOR of this sort may properly be called a QUATERNION, Article 91 ; Pages 89, 90.

§ XVIII. When the factor, q , is regarded (see § VI.) as a GEOMETRICAL QUOTIENT $= \beta \div \alpha = DB \div DA$, it may conveniently be pictured or constructed by a BIRADIAL, ADB , with a curved arrow inserted, and directed from the initial ray DA (the facient, or divisor-line, α), towards the final ray DB (the factum, or dividend-line, β) ; the point D , from which the two rays diverge, is the vertex of the biradial ; a biradial has a SHAPE, or species, depending on the ratio of the lengths of its two rays, and also on the angle which they include ; two biradials may be similar, namely, by their agreeing with each other in these two respects ; but a biradial has also a plane, and an ASPECT, determined by and directed towards that star, or region of infinite space, which the plane may be said to face, and as seen from which the rotation from the initial to the final ray would appear to be positive (right-handed) ; condirectional and contradirectional (or opposite) biradials, included in the class of parallel biradials ; two biradials, which are at once similar and condirectional, are said to be EQUIVALENT BIRADIALS ; examples ; it is proposed to employ (see § XX.) the conception and construction of such biradial figures to assist in determining the conditions of equality between two geometrical quotients, $\beta \div \alpha$, and $\delta \div \gamma$; and also in enumerating the modes of possible inequality, of any two such quotients, Articles 92 to 95 ; Pages 90 to 95.

§ XIX. Analogous determinations for differences of points (see § I.), constructed or pictured by straight lines, with straight arrows attached ; interpretations of the two equations $D - C = B - A$, $D = B - A + C$; D is here the fourth corner of a parallelogram, of which C , A , B are three successive corners, and of which the altitude may vanish ; inversion and alternation of an equation between differences of points, $C - A + B = B - A + C$; vectors are equal, when they differ only in their situations in space ; addition of vectors still corresponds to composition of vections, although they are not now given as successive (compare § v.) ; such addition is commutative and associative, $\alpha + \beta = \beta + \alpha$, $(\gamma + \beta) + \alpha = \gamma + (\beta + \alpha)$; the sum of any set of vectors is simply that one resultant vector which produces the same total or final effect, in changing the position of a point, as all the proposed summand vectors would do, if the motions, of which they are supposed to be the instruments, were simultaneously or successively performed ; the sum of two directed and co-initial sides of a parallelogram is the intermediate and co-initial diagonal ; most of the foregoing results of this section (XIX.) are common to several other modern theories ; a vector (in space) is a species of NATURAL TRIPLET, suggested by geometry, and found to be capable of a triple variety, or to depend upon a system of three distinct elements, which admit of being expressed numerically, and correspond to the TRIDIMENSIONAL character of SPACE ; in the present calculus (compare § XII.), a vector may be represented generally by the TRINOMIAL FORM. $\rho = ix + jy + kz$, where x, y, z are three scalar (or Car-

tesian) co-ordinates, while i, j, k are those three rectangular vector-units, which were introduced (see § x.) in the foregoing Lecture,

Articles 96 to 101; Pages 95 to 105.

- § xx. EQUIVALENT BIRADIALS (see § xviii.) correspond to EQUAL QUOTIENTS; examples; in fact a biradial may be *turned round in its own plane*, or *transported parallel to itself*, or its *legs* may be *altered proportionally*, without changing the *relative direction*, or the *relative length*, of those two legs, or rays, or vectors, and therefore without affecting that *complex* (metrographic) *relation* between the two rays which has been considered (in § vi.) as determining their geometrical *quotient*; hence in this calculus, *as in many other modern systems*, the equation $\delta \div \gamma = \beta \div \alpha$, between two quotients, is interpreted as signifying a *proportionality of lengths*, combined with an *equality of angles in one plane*, between the two pairs of lines, α, β , and γ, δ ; BUT, *when we come to take account of the PLANE OF THE ANGLE*, between any two such lines α, β , and to regard *that plane as VARIABLE IN SPACE*, there arises a NEW DOUBLE VARIETY, in the geometrical quotient $\beta \div \alpha$, or in the numerical elements on which it depends; because we introduce hereby the consideration of the ASPECT (see § xviii.) of the plane, or of the biradial, and thus bring into play (or at least may be conceived to do so) a NEW PAIR OF NUMBERS, such as those which determine in astronomy the *inclination of the plane* of the orbit of a planet or comet to the ecliptic, and the *longitude of its node*, in addition to that FORMER PAIR OF NUMBERS, which determine the *ratio of the lengths* of the two lines compared, and the *magnitude of the angle* between them: the GEOMETRICAL QUOTIENT OF TWO VECTORS is found therefore *again* (compare § xvii.), in this *new* way, by consideration of its *representative biradial*, to involve or depend upon a SYSTEM OF FOUR NUMBERS (*two for shape*, and *two for plane*), and to be (see again § xvii.), in that sense, a QUATERNION, Articles 102 to 107; Pages 106 to 112.
- § xxi. *Multiplication of two arbitrary quaternions*, effected by means of their representative *biradials*, prepared so that the *final* ray of the multiplicand may *coincide* with the *initial* ray of the multiplier, as factum and profaciend; and therefore so that the *identity* $(\gamma \div \beta) \times (\beta \div \alpha) = \gamma \div \alpha$, of § vii., may be employed to form the PRODUCT; this process is absolutely *free from vagueness* in its *conception*, and altogether *definite* in its *results*, which therefore are adapted to become the subject matter of THEOREMS; example, here stated by way of anticipation, $q'' q' \cdot q = q'' \cdot q' q$; this is the *associative principle* of multiplication of quaternions, and will be afterwards fully discussed (in Lectures V., VI., VII.); *Division* of Quaternions may obviously be effected by an entirely analogous process,
- Article 108; Pages 112, 113.
- § xxii. Before entering on the *general* theory of *operations on quaternions*, we may perform operations on *numbers*, and on *lines*, regarded as particular *cases of quaternions*; for example, we can shew that the *tensor of a scalar* is the *absolute* (or arithmetical) *value* of that scalar, $T(\pm 3) = 3$:

and that the *tensor of a vector* is the number expressing the length of that vector, $Ti = Tj = Tk = 1$; $T \cdot \kappa\lambda = T\kappa \cdot T\lambda$, $T(\lambda \div \kappa) = T\lambda \div T\kappa$; $T\rho = \sqrt{-\rho^2}$; $Tw = \sqrt{+w^2}$; it will be proved (in § LXIII.) that generally the tensor of a quaternion q is

$$Tq = T(w + \rho) = \sqrt{(w^2 - \rho^2)};$$

examination and explanation of a formula which may seem at first a paradox, Articles 109 to 112; Pages 113 to 117.

§ XXIII. The *versor of a positive scalar* is the sign +, or the factor + 1; the versor of a *negative scalar* is the sign -, or the factor - 1; the versor $U\rho$, of a vector ρ , is the *vector-unit* in the *direction* of that vector, $U\rho = \rho \div T\rho = \rho \div \sqrt{(-\rho^2)}$, $(U\rho)^2 = -1$; the versor of zero, $U0$, is generally an *indeterminate* symbol, but it may become determinate, if we know, in any particular investigation, the *law* according to which the scalar or vector tends to vanish; a tensor may be treated as a *positive scalar* (instead of a *signless number*); the *conjugate of a scalar* is the scalar itself, but the *conjugate of a vector* is equal to that vector reversed, $Kw = +w$, $K\rho = -\rho$; it may be remarked by anticipation, that the *conjugate of a quaternion* is, generally, see § LXIII.,

$$Kq = K(w + \rho) = w - \rho, \quad . \quad . \quad . \quad . \quad . \quad .$$

Articles 113, 114; Pages 118, 119.

§ XXIV. *Powers of vectors*, the *exponents* being still *scalars*, but the *vector bases* being *not* now *unit-lines* (compare § XIV.); such powers are *quaternions*; examples: the *tensor of the power* is the *power of the tensor*, and the *versor of the power* is the *power of the versor*; $T \cdot \rho^t = (T\rho)^t = T\rho$, $U \cdot \rho^t = (U\rho)^t = U\rho^t$; the power ρ^t , when operating as a factor on a line $\sigma \perp \rho$, produces another line $\tau = \rho^t\sigma$, which also is perpendicular to ρ ; the *direction* of this new line τ is formed from that of σ by a *rotation* through t quadrants round ρ , and its *length* bears to the length of σ a *ratio* expressed by the t^{th} power of the number $T\rho$ which expresses the length of ρ ; the power, or quaternion, or quotient, $\rho^t = \tau \div \sigma$, *degenerates into a scalar* when t is any *even integer*; ρ^0 , for example, is positive unity, and ρ^2 is a negative number, $= -T\rho^2$ (compare §§ XIII., XXII.); on the other hand the power ρ^t degenerates from a quaternion into a *vector*, when the exponent t is any *odd whole number*, for example, $\rho^1 = \rho$; another and more important example is the *reciprocal of ρ* , or the power ρ^{-1} ; *this power is a line*, which, when operating as a factor on a line σ perpendicular to ρ , has the effect of *dividing the length of σ* by the number $T\rho$, and of *causing its direction to turn negatively* (or left-handedly) *through a quadrant*, round ρ as an axis; the tensor and versor of the reciprocal are respectively the reciprocals of the tensor and versor, $T(\rho^{-1}) = (T\rho)^{-1}$, $U(\rho^{-1}) = (U\rho)^{-1} = -U\rho$, $\rho^{-1} = -T\rho^{-1} \cdot U\rho$; any two **RECIPROCAL VECTORS**, ρ and ρ^{-1} , have their **DIRECTIONS OPPOSITE**, and their **LENGTHS RECIPROCAL**; the *product $\beta \times \alpha^{-1}$* is equal to the *quotient $\beta \div \alpha$* , and may be denoted more concisely by $\beta\alpha^{-1}$ or by $\frac{\beta}{\alpha}$, while the re-

reciprocal a^{-1} may also be denoted by $\frac{1}{a}$; for powers of vectors with scalar exponents, we have generally (as in algebra), $\rho^m \rho^n = \rho^{m+n}$,

Articles 115 to 118; Pages 119 to 125.

- § xxv. Illustrations from the *logarithmic spiral*; the *quotient* of two vectors (in space) may generally be put under the *form of a power*, ρ^t , where the *base* ρ is a *vector*, depending (see § XIX.) on a system of *three numbers*, and serving to fix the *aspect and angle of a spiral*; while the *exponent*, t , is (as in § XXIV.) a *scalar*, and serves to mark (in this mode of illustrating the subject) the *fraction of a quadrant at the pole*; the *QUOTIENT of two rays* is therefore *again found*, in this new way, to be a QUATERNION, or to depend generally on a *system of four numerical elements*, Articles 119, 120; Pages 125 to 129.

LECTURE IV.

(Articles 121 to 174; Pages 130 to 185.)

PROPORTIONS OF LINES IN ONE PLANE, POWERS AND ROOTS OF QUATERNIONS; NOTATIONS, $\| \|$, $\angle q$, $\text{Ax. } q$; GEOMETRICAL EMPLOYMENT OF $\sqrt{-1}$, AS A PARTIALLY INDETERMINATE SYMBOL.

- § xxvi. Recapitulation; construction of a *quadrantal quaternion* or of the *quotient of two rectangular lines* (compare § XI.) by a *line* drawn in the *direction of the axis of the versor* of this quotient or quaternion, and with a *length* which represents the *tensor* of the same quadrantal quaternion; thus the *rotation round the quotient-line, from the divisor line to the dividend-line, is positive* (compare again § XI.); examination and confirmation of the *consistency* of this conception of a quotient-line, with *earlier principles* of this calculus; division of one *line* by another (§ VI.) may be regarded, in this view, as a *case* of the division of one *quotient* (§ VII.), or of one *quaternion* (§ XXI.), by *another* quotient or quaternion, but the *results* of these different *views agree*; an *equation* between quotients may in like manner receive *two distinct but harmonizing interpretations*, of which *one* is that (comparatively) *usual one*, referred to in § XX., while the other seems to be peculiar to quaternions,
- Articles 121 to 126; Pages 130 to 139.

- § xxvii. On the same plan *two distinct methods of interpretation* may be applied to the *symbol* $\beta \div \alpha \times \gamma$, where α, β, γ are supposed to be *three coplanar lines*, $\gamma \| \| \alpha, \beta$; but they *both* conduct to *one common line* δ as the *result*, namely, to that fourth line, in the plane of α, β, γ , which is, in *several other systems also*, regarded as the *FOURTH PROPORTIONAL* to those three lines, and satisfies, in a sense already mentioned (§ XX.), the *equation* $\delta \div \gamma = \beta \div \alpha$, or the *proportion* $\alpha : \beta :: \gamma : \delta$, which admits of *inversion* and *alternation*; this proportion gives *two others*, between the *tensors* and the *versors* respectively (see §§ XXII., XXIII.) of the four coplanar

lines; we may write $\delta = \beta\alpha^{-1} \cdot \gamma$, and $\delta = \gamma\alpha^{-1} \cdot \beta$, but are *not yet* entitled to write $\delta = \beta \cdot \alpha^{-1}\gamma$, nor $\delta = \gamma \cdot \alpha^{-1}\beta$, because the *associative principle* of multiplication (compare § XXI.) has not as yet been proved; we may already see that (on the principles above employed) *the fourth proportional to three lines which are NOT coplanar CANNOT BE ANY LINE*; in fact it will be shewn, in the Fifth Lecture, to be a *non-quadrantal quaternion*, Articles 127 to 130; Pages 139 to 144.

§ XXVIII. When the three lines α, β, γ are coplanar, and are supposed to be arranged as the *base*, BC, and the two *successive sides*, CA, AB (following the base), of a *triangle inscribed in a circle*, the fourth proportional δ may be constructed by a certain line AF, which *touches*, at the vertex A, the *segment* BCA (or ACB), or which coincides with the *initial direction* of motion along the circumference, *from A to B, through C*; if a *quadrilateral* ABCD be inscribed in a circle, and if the first side AB be divided by the second side BC, and the quotient multiplied into the third side CD, the resulting line, $DF = AB \div BC \times CD$, will have the direction *opposite* to that of the fourth side DA, or the direction of that fourth side *itself*, according as the quadrilateral is an *uncrossed* or a *crossed* one; the results of *this section* (§ XXVIII.), respecting fourth proportionals to three sides of an inscribed triangle or quadrilateral, do not *essentially* require, for their establishment, any principles *peculiar* to quaternions, Articles 131, 132; Pages 144 to 148.

§ XXIX. The *THIRD PROPORTIONAL* to any two lines α, γ is easily constructed, as a third line ϵ , coplanar with them; but when we have thus the proportion $\alpha : \gamma :: \gamma : \epsilon$, we must NOT generally, in the *present calculus*, write the usual algebraic *equation between square and product*, $\gamma^2 = \alpha\epsilon$, nor $\gamma^2 = \epsilon\alpha$; in fact these two equations are *equally true* in algebra, and in several modern geometrical systems, but $\alpha\epsilon$ is *not generally equal* to $\epsilon\alpha$ in quaternions, on account of the generally *non-commutative* character of multiplication (see §§ X., XI., XV.); we may however write, under the conditions supposed, $\epsilon\alpha^{-1} = (\gamma\alpha^{-1})^2$, $\alpha\epsilon^{-1} = (\gamma\epsilon^{-1})^2$, if we *retain*, for quaternions generally, the *notation* $q^2 = q \times q$, with the corresponding *definition* of a *square*; in like manner we must *not* write, in this calculus, as a general expression for a *MEAN PROPORTIONAL*, $\gamma = \pm \sqrt{\alpha\epsilon}$, but may write $\gamma = \pm (\epsilon\alpha^{-1})^{\frac{1}{2}} \alpha$, in which expression it is proposed to take the *upper sign*, when γ *bisects the angle itself* between the directions of α and ϵ , but the *lower sign* when it *bisects the supplement* of that angle; *is opposite to the bisector* the former of these two cases, γ may be said to be by eminence *THE MEAN* proportional between α and ϵ , its length being also a mean between theirs; *the mean* between two given vectors is thus *in general a determined* vector; but when the two vectors have *opposite* directions, their mean proportional may then take *any direction in the plane perpendicular* to the extremes, Articles 133, 134; Pages 148 to 151.

§ XXX. Analogous interpretations of the two symbols $(\beta\alpha^{-1})^{\frac{1}{2}} \alpha$, $(\beta\alpha^{-1})^{\frac{3}{2}} \alpha$, as denoting the *SIMPLEST PAIR of mean proportionals*, inserted between α and β ; these two means must *not*, in the present calculus, be denoted ge-

nerally by the symbols, $\beta^{\frac{1}{3}} a^{\frac{1}{3}}$, $\beta^{\frac{2}{3}} a^{\frac{1}{3}}$; the tensor and versor of the cube root of a quaternion may be regarded as being respectively the cube-roots of the tensor and the versor; in general we may interpret the POWER q^t of any quaternion q , with any scalar exponent t , as being a new quaternion, of which the tensor and the versor are respectively the same (t^{th}) powers of the tensor and the versor of the old or given quaternion, which is proposed as the BASE of the power; thus (compare § XXIV.),

$$T. q^t = (Tq)^t = Tq^t, \quad U. q^t = (Uq)^t = Uq^t;$$

and we may conceive that this latter power of a versor is itself another versor, which has the effect of turning any line a , in a plane perpendicular to the axis of Uq , or of q , through an angle, or amount of rotation, positive or negative, represented by the product $t \times \angle q$; but in order to develop and apply this general conception, we must first fix definitely what is to be understood in general by the ANGLE, or amplitude, $\angle q$, of a quaternion, or of a versor, Articles 135, 136; Pages 151 to 153.

§ XXXI. If we allow this amplitude $\angle q$ to take any one of the values included in the formula $\angle q = \hat{q} + 2l\pi$, where \hat{q} denotes an Euclidean angle, $\hat{q} > 0$, $< \pi$, we shall then have two values for a square root, three for a cube root, &c., as in the usual theory of roots of unity, and as in those modern geometrical systems which represent all such powers or roots by lines, whereas with us they are quaternions; examples: this view of $\angle q$ would give $\angle (q^t) = t\hat{q} + 2(t+l)\pi$, $\angle (q^u) = u\hat{q} + 2(mu+m')\pi$, $\angle . q^{u+t} = (u+t)\hat{q} + 2p(u+t)\pi + 2p'\pi$, $\angle (q^u \cdot q^t) = (u+t)q + 2(lt+mu+n)\pi$; and in order that we should have generally $q^u q^t = q^{u+t}$, it would be necessary and sufficient to assume $p = m = l$, or, in other words, we should assume one common value $\hat{q} + 2l\pi$ for $\angle q$, in forming the three powers here compared; and after making this assumption, it would still be necessary, in general, to retain that value $t(\hat{q} + 2l\pi)$ of the power q^t , which was immediately given by the multiplication $t \times \angle q$, and not to add to this product any multiple $2l'\pi$ of the circumference, before proceeding to form, by a second multiplication, the angle of the power of a power of a quaternion, if we wish that this new power shall satisfy generally the equation $(q^t)^u = q^{tu}$, Articles 137 to 147; Pages 153 to 163.

§ XXXII. But for the sake of avoiding as much as possible all multiplicity of value of elementary symbols, it appears convenient to define that the notation $\angle q$ shall represent the simplest value of the angle, or that one which most conforms to ordinary geometrical usage, namely, the angle in the first positive semicircle, which was lately denoted by \hat{q} , admitting however 0 and π as limits, and therefore writing $\angle q \geq 0, \leq \pi$; so that the prefixed mark \angle comes to be the characteristic of a definite operation, which may be said to be the operation of TAKING THE ANGLE of any proposed quaternion q ; this view agrees with our earlier definitions (§§ XIV., XXIV.) respecting powers of vectors, and gives $\angle \rho = \frac{\pi}{2}$, so that the angle

of a vector is a right angle ; the angle of a positive scalar is zero, and the angle of a negative scalar is two right angles ; with the single exception of powers of negatives (for which powers, as well as for their bases, the axes are indeterminate), the same definition assigns a determinate quaternion as the value of the t^{th} power of any proposed quaternion q ; and the equation $q^u q^t = q^{u+t}$ is satisfied, each member representing a quaternion, of which the versor has the effect of turning a line perpendicular to the axis of q through an amount of rotation represented by $(u + t) \angle q$,

Articles 148 to 150 ; Pages 163 to 166.

§ xxxiii. On the other hand, although the ROTATION produced by the operation of the power q^t is now correctly and definitely expressed by the product $t \times \angle q$, yet because this product is not generally confined between the limits 0 and π , we cannot now consider it as being generally equal to the angle of the power, because we have agreed (in § xxxii.) to confine the ANGLE of every quaternion, and therefore of the power q^t among the rest, within those limits ; thus with the present DEFINITE SIGNIFICATION of the mark \angle , we must not write generally $\angle (q^t) = t \times \angle q$, but rather $\angle (q^t) = 2n\pi \pm t \angle q$, the axis of the power being in the same direction as the axis Ax . q of the base, or else in the opposite direction, according as it becomes necessary to take the upper or the lower sign ; the square root, $q^{\frac{1}{2}}$, of a (non-scalar) quaternion is acute-angled, and so are the cube-root, $q^{\frac{1}{3}}$, &c., while the axes of these roots coincide with the axis of their common power ; but the square q^2 of an obtuse-angled quaternion q has its angle $\angle (q^2)$ equal to the double of the supplement of the obtuse angle $\angle q$, and has its axis in the direction opposite to that of the axis Ax . q ; with this definite view of powers and roots, although three distinct quaternions may have one common cube, yet only one of them is (by eminence) the cube-root of that cube ; examples : in like manner the symbol $(q^2)^{\frac{1}{2}}$ denotes now definitely $+q$, or $-q$, according as the angle of q is acute or obtuse ; $(\rho^2)^{\frac{1}{2}}$ denotes a vector, with a length = $T\rho$, but with an indeterminate direction, because ρ^2 is a negative scalar ; we must not now write generally $(q^t)^u = q^{ut}$, but may establish this modified formula, $(q^t)^u = (\text{Ax} . q)^{4nu} . q^{ut}$, Articles 151 to 161 ; Pages 166 to 174.

§ xxxiv. Reciprocals and conjugates of quaternions (compare §§ xxiv., xxx.) :

$$\begin{aligned} T(q^{-1}) &= (Tq)^{-1} = Tq^{-1}, \quad U(q^{-1}) = (Uq)^{-1} = Uq^{-1} ; \\ \angle(q^{-1}) &= \angle q, \quad \text{Ax} . (q^{-1}) = -\text{Ax} . q ; \quad Uq^{-1} = KUq = \text{reversor} ; \\ &\quad \angle KUq = \angle Uq, \quad \text{Ax} . KUq = -\text{Ax} . Uq ; \\ &\quad \angle Kq = \angle q, \quad \text{Ax} . Kq = -\text{Ax} . q, \quad TKq = Tq ; \end{aligned}$$

the reciprocal and conjugate of q may be thus expressed,

$$q^{-1} = Tq^{-1} . KUq, \quad Kq = Tq . Uq^{-1} ;$$

in general $qKq = Tq^2$, so that the product of any two conjugate quaternions is a positive scalar, namely, the square of their common tensor ; $Tq = (qKq)^{\frac{1}{2}}$, $Uq = \pm (q \div Kq)^{\frac{1}{2}}$, according as $\angle q < \frac{\pi}{2}$; exam-

ples; when q is a vector $= \rho$, so that $\angle q = \frac{\pi}{2}$, then $Kq = -q$ (compare § XXIII.); and although $(q \div Kq)^{\frac{1}{2}}$ is in this case an *indeterminate vector-unit*, yet we have still $Uq^2 = q \div Kq$, each member being $= -1$, . . .

Articles 162 to 165; Pages 175 to 178.

§ XXXV. More close examination of the CASE OF INDETERMINATION, mentioned in several recent sections, when the base of a power becomes a negative scalar; $\angle (-1) = \pi$; $\text{Ax} \cdot (-1)$ is indeterminate; the symbol $(-1)^t$ or $(-)^t$ denotes a *versor*, which has the effect of producing a given and *definite amount of rotation* $= t\pi$, but in a wholly *arbitrary plane*; in particular, $\angle (-1)^{\frac{1}{2}} = \frac{\pi}{2}$, so that $(-1)^{\frac{1}{2}}$ or $\sqrt{-1}$ represents in this theory (compare §§ X., XXIX., XXXII., XXXIII.) a *quadrantal versor* with an *arbitrary axis*, and therefore also a VECTOR-UNIT with an INDETERMINATE DIRECTION; this *perfectly REAL* but *partially INDETERMINATE INTERPRETATION*, of the symbol $\sqrt{-1}$, is one of the *chief PECULIARITIES* of the present calculus, so far as its connexion with *geometry* is concerned; examples of its *use*, in forming certain EQUATIONS OF LOCI; if o be *origin* of vectors, and P a point upon the *unit-sphere*, then the vector of that point may be expressed as follows:

$$P - O = \rho = \sqrt{-1},$$

so that $\rho^2 + 1 = 0$ is a form for the *equation of a spheric surface*; this form is extensively useful in researches of spherical geometry; the expression $\rho = \beta + b\sqrt{-1}$ represents the vector of a point upon *another sphere*, whose radius is b , and the vector of whose centre is β ; the equation of this new sphere may also be thus written,

$$(\rho - \beta)^2 + b^2 = 0, \text{ or thus, } T(\rho - \beta) = b;$$

the equation $\rho a^{-1} = \sqrt{-1}$, or $(\rho a^{-1})^2 = -1$, may be interpreted as representing a *circular circumference*, namely, the great circle in which the plane through o , perpendicular to a , cuts the sphere which has the origin o for its centre, and has its radius $= Ta$; the indefinite *plane* of the same circle may be represented by the equation $U \cdot \rho a^{-1} = \sqrt{-1}$, and a *parallel plane* by $U \cdot (\rho - \beta) a^{-1} = \sqrt{-1}$; the equation $\rho a^{-1} = (-1)^{\frac{1}{2}}$ represents *another circle*, namely, the *locus of the summits of all the equilateral triangles* which can be described upon the given base a ; and the equation $U \cdot \rho a^{-1} = (-1)^{\frac{1}{2}}$ represents a *sheet of a right cone*, with its vertex at the origin, and with the last-mentioned circle as its base, . . .

Articles 166 to 174; Pages 178 to 185.

LECTURE V.

(Articles 175 to 250; Pages 186 to 240.)

ASSOCIATIVE PRINCIPLE FOR THE MULTIPLICATION OF THREE LINES IN SPACE; QUATERNION VALUES OF THEIR TERNARY PRODUCTS, $\beta\alpha\gamma$, AND FOURTH PROPORTIONALS, $\beta\alpha^{-1}\gamma$; VALUES OF ijk , kji ; GENERAL CONSTRUCTION FOR THE PRODUCT OF TWO VERSORS, BY A TRANSVECTOR ARC UPON A SPHERE.

§ xxxvi. Proof that for any *three coplanar vectors*, α, β, γ , the product $\beta \cdot \alpha^{-1}\gamma$ represents the *same fourth line* δ in their plane as the product $\beta\alpha^{-1} \cdot \gamma$; thus $\beta \cdot \alpha^{-1}\gamma = \beta\alpha^{-1} \cdot \gamma$, at least when $\alpha \parallel \beta, \gamma$ (this last restriction is afterwards shewn to be unnecessary); the proof is given for the three cases, 1st, when the product $\alpha^{-1}\gamma$ is a vector; 2nd, when it is a scalar; and 3rd, when it is a quaternion; in treating these cases, we avail ourselves of the formulæ, $\alpha^{-1} \cdot \alpha\epsilon^{-1} = \epsilon^{-1}$, $\gamma\epsilon \cdot \epsilon^{-1} = \gamma$, $\zeta\eta \cdot \eta^{-1}\theta = \zeta\theta$, which are indeed *included* in the general *associative* principle of multiplication (stated by anticipation in § XXI.), but can be *separately* and more *easily* proved; in general, by the *conceptions* of *reciprocal* and *product*, it can easily be shewn that for any two quaternions q and r , we have, as in algebra, the identities, $r^{-1} \cdot rq = q$, $rq \cdot q^{-1} = r$; another general formula for the multiplication of any two quaternions is $\mu\lambda^{-1} \cdot \lambda\kappa^{-1} = \mu\kappa^{-1}$,
Articles 175 to 182; Pages 186 to 192.

§ xxxvii. *Negatives* of quaternions,

$$T(-q) = Tq, \quad \angle(-q) = \pi - \angle q = \pi - \angle Kq, \quad Ax \cdot (-q) = -Ax \cdot q = Ax \cdot Kq;$$

the *axes* of the negative and conjugate *coincide*, but their angles are *supplementary*;

$$T(-Kq) = Tq, \quad \angle(-Kq) = \pi - \angle q, \quad Ax \cdot (-Kq) = Ax \cdot q;$$

the *negative of the conjugate* has the effect of turning the line on which it operates, round the same axis as the original quaternion, but through a *supplementary angle*; (these results are seen at a later stage, to admit of being connected with the form $Tq(\cos + \sqrt{-1}\sin)\angle q$, to which every quaternion q may be reduced, but in which the $\sqrt{-1}$ is regarded as representing a vector-unit, in the direction of $Ax \cdot q$); $KKq = q$, $K^2 = 1$; $K(-q) = -Kq$; if this = $+q$, then q must be a vector, and *vice versa*; the *tensor and versor of a product or quotient of any two quaternions* are respectively the *product or quotient of the tensors and versors*,

$$T \cdot rq = Tr \cdot Tq, \quad U \cdot rq = Ur \cdot Uq,$$

$$T(r \div q) = Tr \div Tq, \quad U(r \div q) = Ur \div Uq;$$

this result is connected with the mutual *independence of the two acts* or

operations of tension and of version ; the conjugate and the reciprocal of the product of any two quaternions are respectively equal to the product of the conjugates, and to the product of the reciprocals, but taken in an inverted order, $K.rq = Kq.Kr$, $(rq)^{-1} = q^{-1}r^{-1}$; if $\delta = \beta\alpha^{-1}.\gamma = \gamma\alpha^{-1}.\beta$ (see § xxvii.), then $\beta.\alpha^{-1}\gamma = K(-\beta).K(\gamma\alpha^{-1}) = -K(\gamma\alpha^{-1}.\beta) = -K\delta = \delta$; the result of the foregoing section, that $\beta.\alpha^{-1}\gamma = \beta\alpha^{-1}.\gamma$, when α, β, γ are three coplanar vectors, is therefore confirmed in this new way, Articles 183 to 193; Pages 192 to 198.

§ xxxviii. The *associative* principle therefore holds for the multiplication of any three *coplanar* vectors, such as the recent lines γ, α^{-1} , and β , with a *partial* validity of the commutative principle also; so that we may dismiss the *point* in the notation, and may write either $\delta = \beta\alpha^{-1}\gamma$, or $\delta = \gamma\alpha^{-1}\beta$; the line δ may still be called (see § xxvii.) the *Fourth Proportional* to α, β, γ , or to α, γ, β ; but it may also be said to be the *continued product* of $\gamma, \alpha^{-1}, \beta$, or of $\beta, \alpha^{-1}, \gamma$; without introducing -1 as an exponent of the middle factor, if $\mu ||| \lambda, \kappa$, we have the following *equation of coplanarity*, $\mu\lambda\kappa = \kappa\lambda\mu$; each of the symbols here equated denotes a *line*, coplanar with the lines κ, λ, μ , which fourth line in their plane may at pleasure be called the fourth proportional to $\lambda^{-1}, \mu, \kappa$, or to $\lambda^{-1}, \kappa, \mu$, or the continued product of κ, λ, μ , or of μ, λ, κ ; $(\lambda^{-1})^{-1} = \lambda$, $(q^{-1})^{-1} = q$; $\beta\alpha\gamma = \alpha^2.\beta\alpha^{-1}\gamma$; and because $\alpha^2 < 0$ (by § xiii.), the *continued product* $\beta\alpha\gamma$ of three coplanar vectors, γ, α, β , has the direction *opposite* to that of the *fourth proportional* to the lines α, β, γ ; the continued product $(A - C)(C - B)(B - A)$ of the three *successive sides*, AB, BC, CA , of any plane triangle ABC , represents by its *length* the *product of the lengths* of those three sides, and by its *direction* the *tangent at A to the segment ABC of the circumscribed circle* (contrast with this the corresponding result in § xxviii.); this construction of a continued product appears to be *peculiar* to quaternions; case where the three points A, B, C are situated on one straight line; if A, B, C, D be the four successive corners of an *uncrossed* and *inscribed quadrilateral*, the continued product $(D - C)(C - B)(B - A)$, of the *three successive sides* AB, BC, CD , is constructed in this calculus by a line which has the direction of the *fourth side*, DA or $A - D$; but the same product represents a line in the direction *opposite* to that of the fourth side, if the quadrilateral be a crossed one; these results also (which may again be contrasted with those of § xxviii.) appear to be peculiar to quaternions; the formula,

$$U.(D - C)(C - B)(B - A) = \pm U(A - D),$$

expresses, in the present calculus, a property which belongs only to plane and *inscriptible quadrilaterals*, Articles 194 to 200; Pages 198 to 203.

§ xxxix. Interpretation of the fourth proportional $\beta\alpha^{-1}.\gamma$, or $\beta \div a \times \gamma$, for the cases where the three lines $\alpha\beta\gamma$ are *not coplanar*, γ *not* $||| \alpha, \beta$, but where a is *perpendicular* either to γ or to β ; for each of these two cases, the *associative* property of multiplication holds, $\beta\alpha^{-1}.\gamma = \beta.\alpha^{-1}\gamma$, and

the *point* may therefore be omitted; but the symbol $\beta a^{-1} \gamma$ does not now represent any line but a quaternion; the symbol $\beta a \gamma$ denotes another quaternion, which is still (as in the last section) $= a^2 \cdot \beta a^{-1} \gamma$; the versors of these two quaternions are negatives of each other, $U \cdot \beta a \gamma = -U \cdot \beta a^{-1} \gamma$; for any multiplication of any number of quaternions, the tensor of the product is equal to the product of the tensors (compare § xxxvii), $T\Pi = \Pi T$; in the case where the three lines $a\beta\gamma$ compose a rectangular system, the fourth proportional $\beta a^{-1} \gamma$ degenerates from a quaternion to a scalar, which is a negative or a positive number, according as the rotation round a from β to γ is of a positive or a negative character; on the contrary, the continued product $\beta a \gamma$ is positive in the first of these two cases, and negative in the second; thus $\beta a \gamma = -\gamma a \beta = \pm T\beta \cdot Ta \cdot T\gamma$, if $\beta \perp a$, $\gamma \perp a$, $\gamma \perp \beta$, the upper sign holding when the rotation round γ from a to β is positive; if DA, DB, DC be three co-initial edges of a right solid, then

$$(C-D)(B-D)(A-D) = \pm \text{volume of solid,}$$

according as the rotation round the edge DA from DB towards DC is directed to the right hand or to the left; examples from the unit-cube, $k \dot{-} j \times i = -1$, $kji = +1$, $ijk = -1$, . . . Articles 201 to 210; Pages 203 to 208.

§ XL. More general cases, where α, β, γ are neither coplanar, nor rectangular; each of the two symbols, $\beta a^{-1} \cdot \gamma, \beta \cdot a^{-1} \gamma$, represents a determined quaternion, but it remains to prove (§§ XLII., XLIII.) that these two quaternions are equal; it is sufficient for this purpose to establish the equality of their versors, and therefore the lines a, β, γ may be supposed to be three unit-vectors, OA, OB, OC , terminating at three given points A, B, C on the surface of the unit-sphere (§ xxxv.); the quaternion quotient βa^{-1} becomes then a versor, with AOB for its representative biradial (§ xviii.); and the great-circle arc, AB , which subtends the angle AOB , may be said to be the REPRESENTATIVE ARC of the same quaternion or versor, βa^{-1} ; it is proposed to construct the representative arc of the quaternion $\beta a^{-1} \cdot \gamma$,
Articles 211 to 216; Pages 208 to 212.

§ XLI. Equality of any two versors corresponds to equality of their representative arcs, such ARCUAL EQUALITY being defined to include sameness of direction on the spheric surface, of the VECTOR ARCS compared, so that EQUAL ARCS are always supposed to be portions of one common great circle; but an arc may be conceived to slide or turn, in its own plane (compare § xx.), or on the great circle to which it belongs, without any change of value; constructions for multiplication and division of versors, by processes which may be called addition and subtraction of their representative arcs; if any multiplicand versor q , and any multiplier versor r , be represented by two successive sides KL, LM , of a spherical triangle KLM , the product versor rq will be represented by the base KM of the same triangle; thus versor, proversor, and transversor (see § ix.), are represented by what may be called an arcual vector, an arcual provector, and an arcual transversor respectively (compare First Lecture); we may write the formula $\frown LM + \frown KL = \frown KM$, and the ARCUAL SUM of two successive

sides of any spherical triangle, regarded as *two successive vector arcs*, may in this sense be said to be EQUAL TO THE BASE (compare §§ IV., v.); *such ADDITION* (of vector arcs) corresponds to, and represents, a *composition of two successive versions* (§ IX.), or *plane rotations of a line* (the radius); the sum of the *three successive sides* of a spherical triangle, or generally the *sum of all the successive sides of any spherical polygon*, may be said to be a *null arc*, or to be equal to zero, $\frown MK + \frown LM + \frown KL = 0$; to go on the surface of the sphere successively from *K* to *L*, from *L* to *M*, and from *M* to *K* again, produces no final change of position; *SUBTRACTION of vector arcs*, corresponding to *division of versors*, is very easily effected, on the same general plan of construction, and represents (compare again § IX.) a *decomposition of a given version* into two others, of which the *first* in order is given, namely, the one represented by the *subtrahend arc*; in short, for arcs as for lines, the relations of § IV., between vector, provector, and transvector, hold good in this manner of speaking; the *provector arc* is regarded as the *remainder*, in the *arcual subtraction* of vector from transvector; *addition of ARCS is NOT a COMMUTATIVE operation*; for if two arcs *KK'*, *M'M* bisect each other in *L*, we shall have

$$\frown KL + \frown LM = \frown LK' + \frown M'L = \frown M'K',$$

and *this arcual sum* $\frown M'K'$ is indeed *equally long* with the arc $\frown KM$, which was found to be $= \frown LM + \frown KL$, but it is part of a *different great circle*, and therefore these two sums are *not arcually equal* to each other, in the sense of the present section; this result answers to and illustrates the general *non-commutativeness* of the operation of *multiplication of versors*, whereby qr is *not* generally $= rq$ (§§ x., XI., XXIX. &c.); it is necessary to *distinguish* in writing between two such symbols as $\frown + \frown$ and $\frown + \frown'$; the *rule* adopted in this calculus is to write the symbol of the *addend arc*, like that of the *multiplier quaternion*, and generally the **SYMBOL OF THE OPERATOR, to the LEFT of the SYMBOL OF THE OPERAND**, that is, in this case, to the left of the symbol of the arc to which another is to be added; thus we *still* write "provector plus vector," and *not*, generally, vector plus provector; several other general properties of multiplication and division of quaternions may be illustrated by the same method of arcual construction, Articles 217 to 222; Pages 212 to 217.

§ XLII. Application of the method of the last section to the problem proposed at the end of § XL., namely, to the construction of the *representative arc of the fourth proportional* $\beta\alpha^{-1}.\gamma$ to three unit-vectors, α, β, γ , or *OA, OB, OC*, which are *not* rectangular, *nor* in one common plane (§ XL.), but which shall at first be supposed to make *acute* angles with each other, so that the *sides* of the triangle *ABC* shall *each* be *less* than a quadrant; the vector arc representing γ is here a quadrant *KL* with *c* for its positive pole; the provector arc representing the other factor $\beta\alpha^{-1}$, is the arc *AB*, or an equal arc *LM*; the transvector arc *KM*, which represents the required fourth proportional, under the form of the *product* $\beta\alpha^{-1}.\gamma$, is found to have its *pole* at a new point *D*, which is a corner of a new *circumscribed spherical triangle DEF*, whose *sides* *EF, FD, DE* are respec-

tively bisected by the three corners A, B, C of the old or given triangle; and the REPRESENTATIVE ANGLE, KDM, at this pole D, which corresponds to the representative arc, KM, and may replace it, as representing the fourth proportional to the three vectors α, β, γ , is equal to the semisum of the angles of the auxiliary triangle, DEF, or to the supplement of that semisum, according as the rotation round α from β to γ is positive or negative; hence the two quaternions $\beta\alpha^{-1} \cdot \gamma$ and $\gamma\alpha^{-1} \cdot \beta$ have one common axis, namely, the radius OD, but have their angles supplementary; but these were the conditions assigned in § XXXVII., as necessary and sufficient, in order that one quaternion should be the negative of the conjugate of the other; we have therefore, as in the last cited section,

$$\beta\alpha^{-1} \cdot \gamma = -K(\gamma\alpha^{-1} \cdot \beta) = \beta \cdot \alpha^{-1} \gamma,$$

and the associative principle is again found to hold good for the three vectors $\gamma, \alpha^{-1}, \beta$, although these three lines are not now coplanar (as they were in §§ XXXVI., XXXVII.), and do not form a wholly or even partially rectangular system (as they did in § XXXIX.),

Articles 223 to 235; Pages 217 to 228.

§ XLIII. Other proof of the same theorem, by means of an analogous construction for the product $\beta \cdot \alpha^{-1} \gamma$; the case where $\beta \perp \alpha$ may be treated as a limit of a case lately discussed, the arc AB becoming a quadrant, and the triangle DEF becoming a lune; case where the arc AB is greater than a quadrant; value of $\beta\alpha^{-1} \cdot \gamma'$, when $\gamma' = -\gamma$, and when the sides of the new triangle ABC' are each greater than a quadrant; we have

$$\beta\alpha^{-1} \cdot \gamma' = -K(\gamma'\alpha^{-1} \cdot \beta) = \beta \cdot \alpha^{-1} \gamma';$$

in EVERY case, the ASSOCIATIVE PRINCIPLE of multiplication holds good for any system of THREE VECTORS, and we may ALWAYS write in this calculus (as in algebra) the formulæ,

$$\beta \cdot \alpha^{-1} \gamma = \beta\alpha^{-1} \cdot \gamma = \beta\alpha^{-1} \gamma; \quad \beta \cdot \alpha \gamma = \beta\alpha \cdot \gamma = \beta\alpha \gamma;$$

to establish this result has been the main object of the present Lecture, .

Articles 236 to 240; Pages 228 to 233.

§ XLIV. Partial indetermination of the constructed triangle DEF, when the given triangle ABC is triquadrantal; the point D may take infinitely many positions on the sphere, but the semisum of the angles at D, E, F is always equal to two right angles; the scalar character of the fourth proportional to three rectangular vectors, which had been established in § XXXIX., may in this way be proved anew, as a particular or limiting case of a much more general result; when a scalar is treated as a quaternion, its axis is indeterminate; the rule of § XXXIX. for determining the sign of the scalar is also reproduced, Articles 241 to 244; Pages 233 to 237.

§ XLV. Illustrations of the equations (of § XXXIX.), $kji = +1$, $ijk = -1$; the former may be interpreted as expressing that if a line λ be suitably chosen, namely, so as to be perpendicular to the (meridional) plane of k and i , and be then operated on successively by i , by j , and by k , considered as

three quadrantal and mutually rectangular versors (§ x.), the final direction of this revolving line λ will be the same as the initial direction; the latter equation ($ijk = -1$) may in like manner be interpreted as expressing that if the same (westward or eastward) line λ be operated on successively by k , by j , and by i , it will take at last that (eastward or westward) direction which is opposite to the initial direction; and because each of the vector-units i, j, k , when thus regarded as a quadrantal versor, is evidently (see again § x.) a semi-inversor, we have in this way extremely SIMPLE INTERPRETATIONS for ALL THE PARTS OF THE FORMULA,

$$i^2 = j^2 = k^2 = ijk = -1;$$

which continued equation may be considered as including within itself all the laws of the COMBINATION OF THE SYMBOLS, i, j, k ; and therefore ultimately, on the symbolic side, the WHOLE THEORY OF QUATERNIONS, because these are all reducible to expressions of the quadrimomial form,

$$q = w + ix + jy + kz,$$

Articles 245 to 250; Pages 237 to 240.

LECTURE VI.

(Articles 251 to 393; Pages 241 to 380.)

GENERAL ASSOCIATIVE PROPERTY OF THE MULTIPLICATION OF QUATERNIONS; REPRESENTATION OF THE PRODUCT OF TWO VERSORS BY THE EXTERNAL VERTICAL ANGLE OF A SPHERICAL TRIANGLE; CONNEXION OF TERNARY PRODUCTS OF QUATERNIONS WITH SPHERICAL CONICS; CONTINUED PRODUCTS OF THE SIDES OF PLANE OR GAUCHE POLYGONS INSCRIBED IN A CIRCLE OR IN A SPHERE; COMPOSITION OF CONICAL ROTATIONS; THEORY OF SPHERICAL POLYGONS OF MULTIPLICATION, WITH THEIR SYSTEMS OF INSCRIBED CONICS, AND RELATIONS OF FOCAL ENCHAINMENT.

§ XLVI. Postponement of the proof of the distributive principle of the multiplication of quaternions; additional illustrations of the general theory of the fourth proportional to three vectors, which was assigned in the foregoing Lecture; case of coplanarity, regarded as a *limit*,

Articles 251 to 257; Pages 241 to 247.

§ XLVII. The product of the square roots of the successive quotients of the vectors δ, ζ, η , of the corners of a spherical triangle DEF, is a quaternion,

$$q = (\delta\epsilon^{-1})^{\frac{1}{2}} (\epsilon\zeta^{-1})^{\frac{1}{2}} (\zeta\delta^{-1})^{\frac{1}{2}},$$

of which the angle is the semi-excess of the triangle,

$$\angle q = \frac{1}{2} (D + E + F - \pi);$$

and the axis of the same quaternion product has the direction of $\pm \delta$, that

is of OD or of DO, according as the *rotation* round δ from ζ towards ϵ , or that round D from F towards E, is positive or negative,

Articles 258 to 263 ; Pages 247 to 252.

§ XLVIII. General construction for the multiplication of any two quaternions, by a process *analogous to addition* of their REPRESENTATIVE ANGLES (compare §§ XLI., XLII.); if these be made the *base angles* of a spherical triangle, and if the rotation round the vertex of this triangle, from the base angle which represents the multiplier, towards the base angle which represents the multiplicand, be positive, then the PRODUCT is *represented* by the EXTERNAL VERTICAL ANGLE; if we agree to call the *external vertical angle of a spherical triangle* generally the SPHERICAL SUM OF THE TWO BASE ANGLES, when the *positions* of the *vertices* of these several angles *on the sphere* are taken into account, and when the *addend angle* answers to the *multiplier* quaternion, according to the *rule of rotation* above given, we may enunciate a GENERAL RULE for the *multiplication of any two quaternions*, as follows: "the *tensor of the product* is the *arithmetical product of the tensors* (§ XXXVII.), and the *angle of the product* is the *spherical sum of the angles of the factors*;" this new sort of SPHERICAL ADDITION OF ANGLES is connected with a certain *composition of rotations of arcs*; such *addition of angles* (like that of arcs in § XLI.) is a *non-commutative* operation; this result furnishes a new illustration of the non-commutative character of the general multiplication of quaternions; the rotation round the axis or round the *pole* of the multiplier, from that of the multiplicand, towards that of the product (compare §§ XI., XV., XXVI), is always *positive*, . Articles 264 to 272 ; Pages 252 to 261.

§ XLIX. Corollaries from the general construction for multiplication assigned in the foregoing section (XLVIII.); interpretations by it of the symbols $\alpha\beta$, $\beta\alpha^{-1}$, $\beta\alpha^{-1}\beta$, agreeing with the results previously obtained respecting the product, quotient, and third proportional of any two vectors; interpretations of $\beta\frac{1}{2}\alpha\frac{1}{2}$, $\beta\frac{3}{4}\alpha\frac{3}{4}$, $\beta\frac{3}{4}\alpha\frac{1}{2}$, as denoting quaternions (compare §§ XXIX., XXX.); analogous interpretation of the more general symbol $q = \beta^t \alpha^{1-t}$, when α and β are supposed to be unit-vectors; the unit axis $\Delta x \cdot q = OP$, of this quaternion q , describes by its extremity P a curve APB upon the unit-sphere, which curve is the locus of the vertex P of a spherical triangle APB, whose base-angles are complementary; this curve is a *spherical conic*; for any spherical triangle, with α, β, γ for the unit vectors of its corners A, B, C, and with x, y, z for the (generally fractional) numbers of right angles at those corners, the rotation round C from B to A being supposed to be also positive, we have the three equations

$$\gamma^z \beta^y \alpha^x = -1 ; \alpha^x \gamma^z \beta^y = -1 ; \beta^y \alpha^x \gamma^z = -1 ;$$

any one of which will be found to include, when interpreted and developed, by the principles of the present calculus, the *whole doctrine of spherical trigonometry*; with the phraseology recently proposed, the SPHERICAL SUM OF THE THREE ANGLES of any spherical triangle, if taken in a suitable order of succession, is always equal to TWO RIGHT ANGLES,

Articles 273 to 280 ; Pages 261 to 268.

- § L. Interpretation of the symbol rqr^{-1} , where q and r are any two quaternions; this symbol denotes a new quaternion, with the same *tensor*, and same magnitude of *angle*, as the original or *operand* quaternion, q ,

$$T. rqr^{-1} = Tq, \quad \angle. rqr^{-1} = \angle q;$$

but the *axis* of the new quaternion rqr^{-1} is generally *different* from $Ax. q$, and is formed or derived from this latter axis, by a CONICAL and *positive* ROTATION round the axis $Ax. r$, of the other given quaternion, r , through DOUBLE the ANGLE of that quaternion; analogous interpretations of $q^{-1}rq$, $q^t r q^{-t}$; the latter symbol denotes a quaternion formed from r , by making its *axis revolve conically* round the axis of q , through a rotation expressed by the product $2t \times \angle q$; by employing arcs instead of angles, we may interpret the symbol $q (\quad) q^{-1}$, in which q may be said to be the *operating quaternion*, 'as denoting the operation of causing the ARC which represents the *operand quaternion*, and whose symbol is supposed to be inserted within the parentheses, to *move along the DOUBLED ARC* of the operator, without any change of either *length* or *inclination* (like the equator on the ecliptic in precession); if t be still a scalar exponent, $(qrq^{-1})^t = qr^t q^{-1}$; the symbol $q\rho q^{-1}$ denotes a *vector* formed from the vector ρ , and the analogous symbol qBq^{-1} may be used to denote a *body* derived from the body B , by a conical and finite rotation, through $2 \angle q$ round $Ax. q$; to express that this body has *afterwards* been made to revolve through $2 \angle r$ round $Ax. r$, we may employ the following symbol for the *new* position of the body, or system of vectors, $r. qBq^{-1}. r^{-1}$; and so on for *any number of successive and finite rotations*, round *any axes* drawn from or through one *common origin* o ; interpretations of the symbols $q(\alpha + \rho)q^{-1}$, $q(\alpha + B)q^{-1}$; expression for rotation of a body round an axis which does *not* pass through the origin of vectors; symbols $q^{\frac{1}{2}} (\quad) q^{-\frac{1}{2}}$, $\gamma (\quad) \gamma^{-1}$; the former represents a rotation through the *angle itself* of q ; the latter represents a REFLEXION with respect to the line γ , or a *conical rotation* of the operand (whether vector or body), round γ as an axis, through *two right angles*; the formula $\beta. \alpha^{-1} \epsilon \alpha. \beta^{-1} = \beta \alpha^{-1}. \epsilon. \alpha \beta^{-1}$, expresses that *two successive reflexions*, with respect to any two diverging lines α and β , are equivalent upon the whole to a *single conical rotation*, round an axis perpendicular to both those lines, through twice the angle between them,

Articles 281 to 292; Pages 268 to 277.

- § LI. The general demonstration of the *associative* property of the multiplication of *any three quaternions* (mentioned by anticipation in § XXI.), may be made to depend on the corresponding principle for the multiplication of any *three versors*, q, r, s ; when these versors are represented by *arcs* (§ XL.), we may propose to prove that a certain *arcual equation* (§ XLI.) is a *consequence of five other equations* of the same sort; first proof by *spherical conics*; the two partial or *binary products* rq and sr are represented by portions of the two *cyclic arcs* of a conic circumscribed about a *quadrilateral*, whose successive *sides*, or portions of them, represent the three proposed *factors*, q, r, s , and their *ternary product*, srq ; other and more *elementary* geometrical proof of the associative principle, *not intro-*

ducing the conception of a *cone*; second proof by spherical conics; certain *angles* at the *corners* of a *new spherical quadrilateral* ABCD represent the three factors and their total product, while certain other angles at the *foci* EF of an *inscribed conic* represent the two binary products; *three equations* between *spherical angles* are thus shewn to be *consequences* of *three other equations* of the same sort, in such a way as to establish the property above proposed for investigation; it is therefore proved geometrically, in several different ways, that the ASSOCIATIVE PRINCIPLE OF MULTIPLICATION holds good for *any three versors*, and thence for ANY THREE QUATERNIONS, $sr \cdot q = s \cdot rq = srq$; (in the Fifth Lecture this theorem was established only for the multiplication of *any three vectors*); extension to the case of *any number of factors*; *arcual addition* (§ XLII.), and *angular summation* (§ XLVIII.), are also *associative operations*, although they have been seen to be *not generally commutative*,

Articles 293 to 304; Pages 277 to 290.

§ LII. Other forms of the associative principle; if the first, third, and fifth sides of a *spherical hexagon* be respectively and *arcually equal* to the three successive sides of a *spherical triangle*, then the second, fourth, and sixth sides of the same hexagon will be respectively and arcually equal to the three successive sides of *another triangle*; or if the *arcual sum* of three *alternate sides* of a hexagon (fifth plus third plus first) be equal to *zero* (see § XLI.), then the corresponding *sum* of the *three other* alternate sides (sixth plus fourth plus second) will *likewise* vanish; symbolical transformations of the same principle; if $a\delta^{-1} = \gamma\epsilon^{-1}$, then $\zeta\delta^{-1} \cdot a\beta^{-1} = \zeta\epsilon^{-1} \cdot \gamma\beta^{-1}$; if $\delta\epsilon^{-1} = \kappa\lambda^{-1} \cdot \theta\eta^{-1}$, then $\delta\kappa^{-1} = \epsilon\eta^{-1} \cdot \theta\lambda^{-1}$; if $(\epsilon\delta \cdot \gamma\beta) a = \zeta$, then $(a\beta \cdot \gamma\delta) \epsilon = \zeta$; remarks on the necessity that existed for *demonstrating* the *general* associative principle of multiplication, notwithstanding that *to a certain extent* the principle had been previously *defined* to hold good; we may be said to have virtually used the DEFINITIONAL ASSOCIATIVE FORMULA, $rq \cdot a = r \cdot qa$, for the CASE where *a*, *qa*, and *r \cdot qa* were *LINES*, in order to INTERPRET THE PRODUCT, *rq*, of any TWO geometrical *factors*, or *quaternions*; but the very fact of the perfect *definiteness* (§ XXI.) of this *interpretation* of a *binary product* made it *necessary* that we should *not assume but prove* the corresponding formula respecting a GENERAL TERNARY PRODUCT, Articles 305 to 316; Pages 290 to 303.

§ LIII. If the continued product of *any odd number of vectors* be a *line*, it is equal to the product of the *same* vectors, taken in an *inverted order*; and reciprocally, if the continued product of an *odd* number of vectors be *not a line*, it will *not* remain unaltered by such inversion of the order of the factors; on the other hand, if the number of vectors thus multiplied be *even*, the product will be changed to its own *negative*, if it be a *line*, and not otherwise, by such inversion; if the continued product of an *even* number of vectors be a *scalar*, the inversion produces no change; and reciprocally if the continued product of an *even* number of vectors receive no change by inversion of order, that product must be a *scalar*; *conjugates* and *reciprocals* of *products* of *any number of vectors or quaternions*, are

the products of the conjugates or reciprocals of the factors, taken in an inverted order; in § xxxvii. this was only established for the case of *two* factors; the formulæ $K\alpha = -\alpha$, $K.\beta\alpha = +\alpha\beta$ (see §§ xxiii., xv.), may now be extended as follows, $K.\gamma\beta\alpha = -\alpha\beta\gamma$, $K.\delta\gamma\beta\alpha = +\alpha\beta\gamma\delta$, &c., the signs of the results being alternately $-$ and $+$; the construction of § xxxviii., for the continued product of the three sides of an inscribed triangle, may now be extended so as to shew that *the product of the successive sides of a polygon inscribed in a circle is equal either to a scalar, or to a tangential vector, at the first corner of the polygon, according as the number of the sides is even or odd*; thus the continued product of the *four* successive sides of an *inscribed quadrilateral* ABCD is a *scalar*,

$$U.(A-D)(D-C)(C-B)(B-A) = \mp 1,$$

and the upper or lower sign is to be taken, according as the quadrilateral is an *uncrossed* or a *crossed* one (compare §§ xxviii., xxxviii.); this symbolical result appears to be *peculiar* to the present calculus, and contains a *characteristic property of the circle*, corresponding to the known and elementary relations between angles in *alternate segments*, or in the same segment; the *versor of any product of quaternions* is equal to the *product of the versors*, $UII = IUI$, . . . Articles 317 to 322; Pages 303 to 309.

§ LIV. To interpret the continued product of the four sides of a GAUCHE QUADRILATERAL, ABCD, we may conceive it to be *inscribed in a sphere*; the product is a *quaternion*, of which the *axis* has the direction of the outward or inward *normal* to the sphere at the first corner A, according to the character of a certain rotation; the *angle* of the same quaternion product is the angle of the LUNULE, ABCDA, or the angle between the two *small-circle arcs*, ABC, ADC; this includes as a limit the case of a quadrilateral in a *circle*; an analogous construction holds for the continued product of the sides of a GAUCHE HEXAGON, *octagon*, or other polygon with an *even* number of sides, inscribed in a *sphere*; the product is still a quaternion, of which the *axis* is *normal*, or the *plane tangential*, to the sphere, at the first corner of the polygon; construction for the continued product of the sides of a GAUCHE PENTAGON, *heptagon*, &c., inscribed in a sphere; this product is a *tangential vector*, drawn at the first corner; conversely, *if the continued product of the sides of a gauche pentagon ABCDE be a line*, when this product is constructed according to the rules of the present calculus, *the pentagon is inscriptible in a sphere*; hence is derived the following EQUATION OF HOMOSPHERICISM, or condition for five points A, B, C, D, E, being situated upon one common spheric surface,

$$AB \cdot BC \cdot CD \cdot DE \cdot EA = EA \cdot DE \cdot CD \cdot BC \cdot AB;$$

this *vector character* of the product of the sides of a *pentagon in a sphere* includes, as a limit, the *scalar character* of the product of the sides of a *quadrilateral in a circle* (§ LIII.), which latter relation may be expressed by the following EQUATION OF CONCIRCULARITY,

$$AB \cdot BC \cdot CD \cdot DA = DA \cdot CD \cdot BC \cdot AB, \quad . \quad . \quad . \quad . \quad .$$

Articles 323 to 328; Pages 309 to 315.

§ LV. One form of the *equation of the tangent plane* at A to the sphere ABCD is the following :

$$AB \cdot BC \cdot CD \cdot DA \cdot AP = AP \cdot DA \cdot CD \cdot BC \cdot AB ;$$

the two equations,

$$AB \cdot BC \cdot CD \cdot DE \cdot EA = EA \cdot DE \cdot CD \cdot BC \cdot AB,$$

and

$$AB \cdot BC \cdot CD \cdot DA \cdot AE = AE \cdot DA \cdot CD \cdot BC \cdot AB,$$

must therefore be incompatible, except under the supposition that either the point E coincides with A, or that the four points A, B, C, D are coplanar ; in fact when the distributive principle shall have been established (in § LXXV.), it will become clear that the addition of these two equations gives

$$AB \cdot BC \cdot CD \times AE \cdot EA = AE \cdot EA \times CD \cdot BC \cdot AB,$$

and therefore that either

$$AE^2 = 0, \quad AE = 0, \quad E = A,$$

or else

$$AB \cdot BC \cdot CD = CD \cdot BC \cdot AB,$$

which are respectively (compare § XXXVIII.) conditions of coincidence and coplanarity ; problem of inscription in a given sphere, of a *gauche* quadrilateral ABCD, whose four successive sides AB, . . . DA shall be respectively parallel to four given radii OI, OK, OL, OM ; problem of expressing an n^{th} radius, OP_n , or ρ_n , of a given sphere, considered as a function of an initial radius OP or ρ , and of n other radii, $OI_1, \dots OI_n$, or $t_1, \dots t_n$, to which the n successive and rectilinear *chords* $PP_1, \dots P_{n-1}P_n$ are required to be parallel ; if α and β be any two equally long and diverging lines, OA, OB , and if γ have either of the two opposite directions of the lines AB, BA connecting their extremities, then $\beta = -\gamma\alpha\gamma^{-1}$; hence in the recent question, $\rho_1 = -t_1\rho t_1^{-1}$, $\rho_2 = -t_2\rho_1 t_2^{-1}$, &c., and if we introduce the quaternion, $q_n = t_n \dots t_2 t_1$, the solution of the problem will be expressed by the formula $\rho_n = (-)^n q_n \rho q_n^{-1}$; the same expression will hold good, if we regard the quaternion q_n as the continued product

$$q_n = (a_n - \rho_{n-1}) (a_{n-1} - \rho_{n-2}) \dots (a_1 - \rho),$$

of the n first segments PA_1, P_1A_2, \dots &c., of the n successive chords, on which A_1, A_2, \dots , are n points arbitrarily taken, but not supposed to be situated upon the surface of the sphere ; relation to a conical rotation (see § L.); EQUATION OF CLOSURE, $\rho_n = \rho$; for an inscribed and *even-sided polygon*, $\rho q_n = q_n \rho$, $Ax \cdot q_n \parallel \rho$, with inclusion of the limiting case for which the product q_n is a scalar ; for an *odd-sided* polygon, $\rho q_n = -q_n \rho$, and the same product q_n must reduce itself to a vector $\perp \rho$; these last results agree with those of § LIV. ; if, in a sphere, the five successive sides of an *inscribed gauche pentagon*, ABCDE, be respectively parallel to the five radii drawn to the five corners of a *superscribed spherical pentagon*, IKLMN, then the *fifth corner* N of the *second* pentagon is situated somewhere upon that *great circle* FH, of which a portion coincides with the

arcual sum, $\frown LM + \frown IK$ (see § XLI.) of the *first and third sides* of that second pentagon; this theorem involves and expresses a GRAPHIC PROPERTY OF THE SPHERE, which is *sufficient to characterize that surface*, and is *analogous to the well-known and elementary relation between the DIRECTIONS of the sides of a quadrilateral inscribed in a circle*; indeed this graphic property of the *circle* can be derived as a *limit* from the lately stated and graphic property of the *sphere*; theorem respecting a general relation of an inscribed gauche polygon of $2n$ sides, to a certain other inscribed polygon of $4n + 1$ sides; examples,

Articles 329 to 340; Pages 315 to 325.

§ LVI. *Composition of conical rotations*; the symbol $srqB$ (srq) $^{-1}$ denotes the position into which the body B is brought, by *three successive and finite rotations*, round the three successive *axes*, $Ax . q$, $Ax . r$, $Ax . s$, all drawn from the origin o , through the three successive *angles* denoted by $2 \angle q$, $2 \angle r$, $2 \angle s$; but the same final position of the body, or of the system of vectors operated on (compare § L.), can also be attained by a *single resultant rotation*, round $Ax . srq$, through $2 \angle . srq$; in like manner *any number* of successive and conical rotations of a line ρ , or body B, round axes passing through one common point o , can be *compounded* into one, by *multiplying* together, in the given order, the *quaternions* which represent, by their axes and angles, the *halves* of the given rotations, and then taking the axis and the *doubled angle* of the quaternion *product*; examples: the identity $\beta \div a = \beta \times a^{-1}$ of § XXIV., since it gives $(\beta \div a) \rho (a \div \beta) = \beta . a^{-1} \rho a . \beta^{-1}$, may be interpreted (see again § L.) as expressing that two successive reflexions of an arbitrary line ρ , with respect to two given lines a , β , are jointly equivalent to the double of the conical rotation *represented* by the arc AB; the identity, $\gamma \div a = (\gamma \div \beta) \times (\beta \div a)$, of § VII., conducts in like manner to the conclusion that a conical rotation thus represented by the double of an arc AB, if followed by another conical rotation represented by the double of a successive arc BC, produces on the whole the same effect as that third and *resultant conical rotation*, which is on the same plan represented by the double of the arc AC; that is, by THE DOUBLE OF THE ARCUAL SUM (see § XLI.) of the HALVES of the arcs which represent the two component rotations; *three* successive and conical rotations, represented by the *doubles* of the *three successive sides* of any spherical triangle, produce on the whole *no effect*; geometrical illustrations and confirmations of these results; extension to spherical polygons, and to *any number* of successive rotations, represented by the doubles of the sides; rotations may be *represented* also by *spherical angles* (instead of arcs); the equation $\gamma^2 \beta^2 a^2 = -1$, of § XLIX., shews that if the double of the rotation represented by the angle CAB be followed by the double of the rotation represented by the angle ABC, the result will be the double of the rotation represented by the angle ACB, or the *opposite* of the double of the rotation represented by BCA; two successive reflexions, with respect to two *rectangular lines*, are equivalent to a *single reflexion* with respect to a line perpendicular to both; if a body

$$\gamma^2 \beta^2 a^2 = -1$$

be made to revolve through any number of successive rotations, represented as to their axes and amplitudes by the doubles of the angles of any spherical polygon, the body will be thereby brought back to its original position, Articles 341 to 349 ; Pages 325 to 334.

§ LVII. The system of the two successive rotations represented by the two successive sides DF, FE , of any spherical triangle, is equivalent to a single rotation, represented by the double of the arc which is the common bisector of those two sides ; the arcual sum $\frac{1}{2} \frown ED + \frac{1}{2} \frown FE + \frac{1}{2} \frown DF$, of the halves of the three successive sides of any such triangle DEF , is an arc which has the first corner D of that triangle for its positive or negative pole, according as the rotation round D from F towards E is positive or negative ; the length of the same sum-arc represents the spherical semi-excess, or semi-area, of the triangle ; extension to any spherical polygon, and even to ANY CLOSED FIGURE ON A SPHERE ; case of negative areas ; successive rotations, represented by the successive sides of any spherical triangle or polygon (and not now by the doubled sides), or even by the successive elements of any closed perimeter on a sphere, compound themselves into a single resultant rotation round the first corner or point of the figure, or round the radius drawn to it, through an angle which is numerically equal to the TOTAL AREA of the figure (the case of negative elements of area being attended to when necessary) ; if a body, or system of vectors, be made to revolve in succession round any number of different axes, all passing through one fixed point, so as first to bring a moveable line a into coincidence with a fixed line β , by a rotation round an axis perpendicular to both ; secondly, to bring the same moveable line a from the position β to another given position γ , by revolving in a new plane ; and so on, till after bringing it to coincide successively with any number of lines given and fixed, and finally after turning from κ to λ , the line a is brought back from λ to its own original position ; then the BODY will be brought, by this succession of rotations, into the same final position as if it had revolved ROUND THE ORIGINAL POSITION of the moveable line (a), as an axis, through an angle of finite rotation which has the same numerical measure as the SPHERICAL OPENING of the PYRAMID ($\alpha, \beta, \gamma, \dots \kappa, \lambda$), whose edges are the successive positions of the line ; in symbols, for the case of five given lines, including the original position of a , if we form the quaternion product,

$$q = \left(\frac{\alpha}{\varepsilon} \right)^{\frac{1}{2}} \left(\frac{\varepsilon}{\delta} \right)^{\frac{1}{2}} \left(\frac{\delta}{\gamma} \right)^{\frac{1}{2}} \left(\frac{\gamma}{\beta} \right)^{\frac{1}{2}} \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}},$$

and if the rotations round a , from β to γ , from γ to δ , and from δ to ε be positive, then

$$Tq = 1, \quad Ax \cdot q = \alpha, \quad \angle q = \frac{1}{2} (A + B + C + D + E - 3\pi),$$

the addition of the five angles of the pentagon being performed in the usual way (and not here by such spherical summation as was mentioned in § XLVIII.) ; extension to the product of the square roots of any number

of successive quotients of vectors ; even if that number be infinite, this product of square roots is still a definite quaternion, of which the angle represents the semi-area of a closed figure on a sphere, while the axis of this latter product is still the radius drawn to the first point of the figure ; interpretation of the symbols,

$$\frac{\beta}{\alpha} \frac{\gamma}{\beta} \frac{\alpha}{\beta} \frown AB + \frown BC + \frown CA ;$$

if (as in § XLII.) the corners A, B, C of one spherical triangle bisect respectively the sides opposite to the corners D, E, F of another, and if a body be made to revolve in succession through three rotations represented respectively by $2 \frown CA$, $2 \frown BC$, $2 \frown AB$, or by the DOUBLES OF THE THREE SIDES of the first triangle ABC, taken in an INVERTED ORDER, this body will on the whole have revolved round the corner D of the second triangle, as round a NEGATIVE POLE, through an angle which is numerically equivalent to the DOUBLED AREA of the same second triangle, DEF, . . .

Articles 350 to 357 ; Pages 334 to 343.

§ LVIII. New elementary proof of the associative property of multiplication of three quaternions ; six double co-arcualities may be assumed to exist by construction, and then the theorem is, that three arcual equations are consequences of three others ; this corresponds to the second proof by spherical conics in § LI., which shewed that three equations between angles were consequences of three others : if q, r, s, t , be any four given quaternions, and u their total or quaternary product, $u = tsrq$, while v, w, x denote respectively their three binary products, rq, sr, ts , and y, z denote their two ternary products, srq, tsr ; if also these ten factors and products $q, r, s, t, u, v, w, x, y, z$, be represented by ten angles at ten points A, B, C, D, E, F, G, H, I, K upon the unit-sphere, then since $y = sv$, $z = tw$, $u = ty$, we can, by six triangles, answering to six binary multiplications, construct successively the six points F, G, H, I, K, and E, the four points A, B, C, D being here regarded as given, and also certain angles at them ; in this process of construction, $\angle r$ is represented by two different angles at B, giving one equation of condition ; $\angle s$ is represented by three different angles at C, giving two other such equations ; $\angle t$ gives two equations ; $\angle v, \angle w$, and $\angle y$ give each one other equation : but the angles of q, x, z, u , are each only once employed in the construction ; on the whole then there are EIGHT EQUATIONS OF CONSTRUCTION, required for the correctness of the figure ; but the associative principle gives four other binary products, $y = wq, z = xr, u = xv, u = zq$, and four other triangles ; there are thus TEN TRIANGLES in the completed figure, representing ten binary multiplications (on the plan of § XLVIII.), and it is found that each of the ten points A . . . K is a common corner of three of those ten triangles ; at each point three angles are equal, and there are thus as many as TWENTY EQUATIONS between angles, including the eight equations of construction ; the remaining twelve equations are therefore consequences of those eight, in virtue of the associative principle, . . . Articles 358 to 364 ; Pages 343 to 350.

§ LIX. In general, if there be *any number*, n , of quaternions (or versors), $q_1 \dots q_n$, represented by angles at n points, $Q_1, \dots Q_n$ on a sphere, and if the *total product* $q = q_n q_{n-1} \dots q_2 q_1$ be represented at another point Q , we may conceive these points to be the successive corners of a certain spherical polygon of $p = n + 1$ sides, which may be called a **POLYGON OF MULTIPLICATION**; this conception includes the cases of the *triangle of binary multiplication* in § XLVIII., the second *quadrilateral of ternary multiplication*, ABCD, in § LI., and the *pentagon of quaternary multiplication*, ABCDE, in § LVIII.; in general we may form $n - 1$ *binary products*, $r_1 = q_2 q_1$, &c., $n - 2$ *ternary products*, $s_1 = q_3 q_2 q_1$, &c., and so on; the *number* of these intermediate or *partial products*, or of their representative *points* on the sphere, is $\frac{1}{2} (n + 1) (n - 2)$; along with the p former points, they make up altogether $\frac{1}{2} (n + 1) n$ points in the *completed figure*; each point may be supposed to have *two spherical co-ordinates*, but between these $(n + 1) n$ co-ordinates there exist generally $n (n - 2)$ relations, or *equations of condition*, because they are all determined by the n versors $q_1 \dots q_n$, and therefore by $3n$ numbers (compare § XVII.); other proof of the general existence of $n (n - 2)$ equations of condition, or equations between certain angles in the figure; each of the $\frac{1}{2} (n + 1) n$ points of the figure is a common corner of $n - 1$ different *triangles*, respecting so many *binary multiplications*; at each point, $n - 1$ angles are equal, and thus there are in all $\frac{1}{2} n (n + 1) (n - 2)$ equations between angles; of these, $n (n - 2)$ are *true by construction* (as above), and the remaining angular equations are true by the *associative principle*; there are therefore $\frac{1}{2} n (n - 1) (n - 2)$ EQUATIONS OF ASSOCIATION, which are consequences of $n (n - 2)$ EQUATIONS OF CONSTRUCTION; and the *dependent equations* are more numerous than those on which they depend, whenever the *number* n of the proposed factors exceeds three; in the *complete construction* of a *polygon of multiplication*, with $p = n + 1$ corners, and $\frac{1}{2} p (p - 3)$ inserted points (representing *partial products*), is involved (by the associative principle) the construction of a number of *auxiliary spherical polygons* of inferior degree, expressed by the formula $\frac{p (p - 1) (p - 2) \dots (p - p' + 1)}{1 \cdot 2 \cdot 3 \dots p'}$, if p' be the number of sides of the auxiliary and inferior polygon; this result is not to be confounded with the elementary theorem of combinations, expressed by the same formula, . . .

Articles 365 to 378; Pages 351 to 366.

§ LX. The *focal character*, mentioned in § LI., of the points E, F which represent the two *binary products* rq, sr , in any case of *ternary multiplication*, srq , namely, that they are *foci of a spherical conic* inscribed in the quadrilateral ABCD, if A, B, C, D be the four points which represent the *three factors*, q, r, s , and their total or *ternary product*, may be denoted by the formula,

$$EF (.) ABCD,$$

which admits of various transformations; in the complete construction of the p -sided polygon of multiplication, there arises a *system of such conics*,

in number amounting to $\frac{1}{2}p(p-1)(p-2)(p-3)$, and inscribed in so many quadrilaterals; their *foci* are the $\frac{1}{2}p(p-3)$ *inserted points* (of § LX.), which represent the *partial products*; these points may therefore be called the **FOCAL POINTS** of the *polygon of multiplication*; and if they be conceived to be the corners of a certain *other polygon* or polygons, there will exist, between these different polygons, a species of **FOCAL ENCHAINMENT**; examples; table of *fifteen focal relations*, for the case of the general *hexagon of multiplication*; this hexagon is in this way connected or *enchained* with a certain *other hexagon*, and also with a *triangle* on the sphere, the *nine* corners of which *auxiliary* hexagon and triangle are *foci* of a *system of fifteen spherical conics, inscribed in fifteen spherical quadrilaterals* of the completed figure; geometrical and numerical illustrations; the general *pentagon of multiplication* ABCDE (of § LVIII.) is in an analogous way *focally enchained* with *another pentagon* FIGKH (or with FGHK), by a *system of five conics*, giving the five following focal relations:

$$\begin{aligned} &FG (\cdot \cdot) ABCI; GH (\cdot \cdot) BCDE; \\ &HI (\cdot \cdot) CDEF; IK (\cdot \cdot) DEAG; KF (\cdot \cdot) EABH; \end{aligned}$$

each conic has its foci at two corners of the second spherical pentagon, and touches two sides of the first; elementary illustration, taken from the limiting case where the pentagons become *regular* and plane,

Articles 379 to 393; Pages 366 to 380.

LECTURE VII.

ADDITION AND SUBTRACTION OF QUATERNIONS; SEPARATION OF THE SCALAR AND VECTOR PARTS; NOTATIONS \mathbf{S} AND \mathbf{V} ; DISTRIBUTIVE PRINCIPLE OF MULTIPLICATION OF QUATERNIONS; NEW PROOF OF THE ASSOCIATIVE PRINCIPLE; GEOMETRICAL APPLICATIONS OF THESE PRINCIPLES, INCLUDING SOME NEW GENERATIONS AND PROPERTIES OF THE ELLIPSOID; NEW REPRESENTATIONS OF LOCI; CONNEXIONS OF QUATERNIONS WITH CO-ORDINATES, DETERMINANTS, TRIGONOMETRY, LOGARITHMS, SERIES, LINEAR AND QUADRATIC EQUATIONS, DIFFERENTIALS, AND CONTINUED FRACTIONS; INTRODUCTION OF THE BIQUATERNION.

§ LXI. Recapitulation, Articles 394 to 400; Pages 381 to 386.

§ LXII. *Addition of a number to a line*; interpretation of the symbol $1+k$; we look out for some *common operand*, that is, for some *one line* such as i , on which the two proposed summands, k and 1 , can *both* operate separately as *factors*, in ways already considered, so as to produce two separate results or *partial products*, which shall themselves be or denote *lines*, namely, in this case j and i ; we then *add these two lines* (§§ v., XIX.), so as to form a *new line* $(i+j)$; finally we *divide the sum by the common operand*, and we take the *quotient* $(i+j) \div i$, obtained by this division,

which *quotient* is in general (see §§ VI., XX.) a QUATERNION, as the value of the proposed SUM,

$$1 + k = (1i + ki) \div i = (i + j) \div i;$$

the effect of $1 + k$, as a factor, is to change the *side* of a horizontal square to that *diagonal* of the same square which is more advanced than it in azimuth by 45° ;

$$T(1 + k) = 2^{\frac{1}{2}}, \quad U(1 + k) = k^{\frac{1}{2}}, \quad 1 + k = 2^{\frac{1}{2}} k^{\frac{1}{2}};$$

this plan of *interpretation* of the symbol $1 + k$ is analogous to that employed in the calculus of finite differences for the interpretation of the symbol $1 + \Delta$, in which *also* the two summands appear at first as *heterogeneous*, but are *incorporated* by being made to operate on one *common function* fx ; more elementary illustration of the process; in general the symbol $w + \rho$, where w denotes a scalar, and ρ a vector, can on the same plan be interpreted as a *quotient of two lines*, and therefore as a *quaternion*, by taking some line $a \perp \rho$, and defining that $w + \rho = (wa + \rho a) \div a$, when wa and ρa are *lines*; addition of this sort is a perfectly *definite* operation, and has the *commutative* character, $w + \rho = \rho + w$,

Articles 401 to 405; Pages 387 to 391.

§ LXIII. Conversely, an arbitrary quaternion q can always be *definitely decomposed* into *two parts*, such as w and ρ , of which one shall be a *number* and the other a *line*, although it is possible that one of these parts may vanish; if $q = \beta \div a$, and if we *decompose the dividend line* β by *projection* into two *partial vectors*, or summand lines, β, β' , respectively *parallel* and *perpendicular* to the divisor line a , and divide *each part* separately by that line a , the partial quotients thus obtained will be respectively *the scalar part* and *the vector part* of the *total quotient* or quaternion q ; introducing then the letters S and V, as *characteristic of the two operations* of TAKING THE SCALAR and TAKING THE VECTOR of a quaternion, we shall have $S(w + \rho) = w$, $V(w + \rho) = \rho$, and $S(\beta \div a) = \beta' \div a$, $V(\beta \div a) = \beta'' \div a$, if $\beta = \beta' + \beta''$, $\beta' \parallel a$, $\beta'' \perp a$; $q = Sq + Vq = Vq + Sq$, $1 = S + V = V + S$; also (compare § XVI.), $S^2 = S$, $SV = VS = 0$, $V^2 = V$; thus, $Sw = w$, $S\rho = 0$, $Vw = 0$, $V\rho = \rho$; *conjugate quaternions* have *equal scalars* but *opposite vectors*, $SKq = +Sq$, $VKq = -Vq$, $SK = S$, $VK = -V$; $K(w + \rho) = w - \rho$ (§ XXIII.); $Kq = Sq - Vq$, $K = S - V$; $TK = T$ (§ XXXIV.), $T(w + \rho) = T(w - \rho) = (w^2 - \rho^2)^{\frac{1}{2}}$ (§ XXII.); if x be a scalar, $Vx = 0$, then $S \cdot xq = xSq$, $V \cdot xq = xVq$; for example,

$$\begin{aligned} S(-q) &= -Sq, & V(-q) &= -Vq; \\ S(-Kq) &= -Sq, & V(-Kq) &= +Vq, & -K &= V - S; \\ x(w + \rho) &= xw + x\rho; & STq &= +Tq, & VTq &= 0; \\ Sq = Tq \cdot SUq, & Vq = Tq \cdot VUq; & VUq &= UVq \cdot TVUq; \\ UVq &= Ax \cdot q, & (UVq)^2 &= -1, & UVq &= \sqrt{-1}; \end{aligned}$$

quaternions are connected with trigonometry, by the relations,

$$SUq = \cos \angle q, \quad TVUq = \sin \angle q;$$

these reproduce the following general expression of well-known *form*, as representing in this system the *versor* of a quaternion,

$$Uq = SUq + VUq = \cos \angle q + \sqrt{-1} \sin \angle q;$$

but the symbol $\sqrt{-1}$ here denotes (compare § XXIII.) the *particular vector-unit* which is drawn in the direction of UVq or of $Ax \cdot q$, that is, in the direction of the axis of the versor; the *indetermination* mentioned in the Fourth Lecture (§ XXXV.) thus disappearing, when Uq is a *determined versor*, Articles 406 to 411; Pages 391 to 397.

§ LXIV. *Expressions for GEOMETRICAL LOCI*, supplied by the symbols S and V ; the *scalar* of a quaternion is positive, null or negative, according as the *angle* of the quaternion is acute, right, or obtuse; $S(\rho \div a) = S \cdot \rho a^{-1} \geq 0$, according as $\hat{a}\rho \leq \frac{\pi}{2}$, if the symbol $\hat{a}\rho$ here denote the *angle between the directions* of the two lines a, ρ , and therefore the *angle of their quotient*, regarded as a *quaternion* (but *not* the angle of that *other* quaternion which is their *product*); to write the equation $S(\rho \div a) = 0$, or $S \cdot \rho a^{-1} = 0$, is therefore to express, by the notations of this calculus, that the line ρ is perpendicular to the line a , and consequently that the locus of the point P is a *PLANE through the origin O, perpendicular to the given line OA*, if $a = OA$, $\rho = OP$; if also $\beta = OB$, the equation $S \cdot (\rho - \beta) a^{-1} = 0$ expresses the perpendicularity $\rho - \beta \perp a$, and gives, as the locus of P , a plane through B , perpendicular to OA , or parallel to the former plane; such a *parallel plane* may also be denoted by the equation $S \cdot \rho a^{-1} = a$, where the scalar a is such that aa denotes the constant projection $\rho' = OP'$ of the variable vector ρ on the fixed vector a ; the equation $S \cdot \rho a^{-1} = 1$ expresses that the projection of a on ρ is the line ρ itself, or that the angle OPA is right; it gives, therefore, as the locus of P , a *SPHERE with OA for diameter*; the *same* spheric surface may also be denoted by either of the equations,

$$S \cdot (a - \rho) \rho^{-1} = 0, \quad T \left(\rho - \frac{a}{2} \right) = \frac{1}{2} T a;$$

methods of *transforming*, by calculation, any one of these *equi-significant forms* into any other, will be explained at a later stage (in § LXXVI.); more generally the two equations,

$$T \left\{ \rho - \frac{1}{2} (a + \beta) \right\} = T \left\{ \frac{1}{2} (a - \beta) \right\}, \quad S \frac{a - \rho}{\rho - \beta} = 0,$$

each represent a sphere described on AB as diameter,

Articles 412 to 415; Pages 397 to 402.

§ LXV. The system of the two equations $S \cdot \rho a^{-1} = 1$, $S \cdot \beta \rho^{-1} = 1$, represents a *CIRCLE*, namely, the mutual intersection of the *plane* through A , perpendicular to OA , and the *sphere* on OB , as diameter; the *product* of the same two equations, namely, the equation $S \cdot \rho a^{-1} \cdot S \cdot \beta \rho^{-1} = 1$, represents a *CONE*, with the last described circle for its *base*; if this last

equation be combined with the equation of a *new plane*, $S \cdot \rho\gamma^{-1} = 1$, the resulting system represents a **PLANE CONIC**, considered as a *curve in space*; the equation of the cone may also be thus written,

$$S \frac{\rho}{\beta^{-1}} S \frac{\alpha^{-1}}{\rho} = 1;$$

under this form it gives the **SUBCONTRARY CIRCULAR SECTION** of the cone, namely, as the intersection of the sphere described on α^{-1} as diameter, with the plane $S \cdot \rho\beta = 1$; the *parallel plane through the vertex*, $S \cdot \rho\beta = 0$, touches the former sphere $S \cdot \beta\rho^{-1} = 1$, which contained the former circular base; this latter plane, and the plane $S \cdot \rho\alpha = 0$, are the **TWO CYCLIC PLANES** of the cone; the equations of these two planes may also be thus written, $S \cdot \beta\rho = 0$, $S \cdot \alpha\rho = 0$; for in general (by §§ XV., LXIII.), $S \cdot \rho\alpha = SK \cdot \rho\alpha = S \cdot \alpha\rho$; thus, in taking the *scalar of the product of any two vectors*, we are allowed to *alter their order*; more generally it will be found (see § LXXXIX.), that *under the sign S we may alter CYCLICALLY the ORDER of any NUMBER of factors*, even if those factors be *quaternions*; a **SPHERICAL CONIC** may be expressed by combining either of the two forms above assigned for the equation of the cone with any one of the three following forms for the equation of the **CONCENTRIC SPHERE**,

$$T\rho = c, \rho^2 + c^2 = 0, S \frac{\rho - \gamma}{\rho + \gamma} = 0;$$

γ is here the vector of some one point upon the sphere, and c is the length of the radius; we might also represent the same concentric sphere by the equation $T\rho = T\gamma$, or $\rho^2 = \gamma^2$; one **CYCLIC ARC** may be represented by the two equations $S \cdot \alpha\rho = 0$, $T\rho = c$, and the *other cyclic arc* by the equations, $S \cdot \beta\rho = 0$, $T\rho = c$, Articles 416 to 421; Pages 402 to 407.

§ LXVI. If a given sphere with a for radius have its centre at the origin o , and if we conceive τ to be a sought point of contact of the sphere with a rectilinear tangent from a given external point s , and make $\sigma = os$, $\tau = o\tau$, we shall have the two equations $\tau^2 = -a^2$, $S \cdot \sigma\tau^{-1} = 1$, the first denoting the *given* sphere round o , and the second an *auxiliary* sphere on os ; the **POLAR PLANE** of the point s , or the plane of which s is the **POLE**, with respect to the given sphere, is the plane of the circle of intersection of the two spheres, and its equation (obtained by suitably multiplying *their* equations) is $S \cdot \sigma\tau = -a^2$, or $S \cdot \tau\mu^{-1} = 1$, if we make $\mu = oM = -a^2\sigma^{-1}$; τ is here treated as a variable vector, but σ and μ as fixed vectors; $U\mu = U\sigma$, $T\mu = a^2T\sigma^{-1}$; M is the *centre of the circle of contact* of the given sphere with the **ENVELOPING CONE** of tangents drawn from S ; if $\rho = oP$ be the variable vector of a point P upon this cone, then

$$\{(S \cdot \sigma(\rho - \sigma))\}^2 = (\sigma^2 + a^2)(\rho - \sigma)^2;$$

but a simpler form of the *equation of the enveloping cone* will be assigned afterwards (in § LXXVII.); the cone which cuts this enveloping cone perpendicularly along the above-mentioned circle of contact, and has its ver-

tex at the centre of the given sphere, is $(S \cdot \sigma\rho)^2 + a^2\rho^2 = 0$; the equation $S \cdot \sigma\rho = -a^2$ expresses that the points P and s are CONJUGATE POINTS, with respect to the given sphere; the equations $S \cdot \rho\sigma = -a^2$, $S \cdot \rho'\sigma' = -a^2$, represent jointly a RIGHT LINE, which is the POLAR of the line ss' ; the continued equation,

$$S \cdot \rho\sigma = S \cdot \rho'\sigma' = S \cdot \rho'\sigma = S \cdot \rho\sigma' = -a^2,$$

expresses that the two lines PP' , ss' , are RECIPROCAL POLARS of each other, with reference to the same given sphere as before; in general, for any two vectors ρ and σ ,

$$S \cdot \rho\sigma = T\rho T\sigma \cos(\pi - \hat{\rho}\sigma);$$

the scalar of the product of any two lines is equal to the rectangle under the lines, multiplied by the cosine of the supplement of the angle between their directions; $\angle \cdot \rho\sigma = \pi - \hat{\rho}\sigma = \pi - \angle \cdot \rho\sigma^{-1}$;

$$SU \cdot \rho\sigma^{-1} = + \cos \hat{\rho}\sigma, \quad SU \cdot \rho\sigma = - \cos \hat{\rho}\sigma;$$

this supplementary relation between the angles of the product and quotient of two lines (compare § LXIV.), is one which it is important to remember in this calculus, from the principles of which it was deduced so early as in § XV.; it may also be considered as connected with the negative character of the square of a vector (§ XIII.), since $\beta\alpha = a^2 \cdot \beta\alpha^{-1} = -T a^2 \cdot \beta\alpha^{-1}$, $U \cdot \beta\alpha = -U \cdot \beta\alpha^{-1}$, and the angle of the negative of a quaternion is the supplement (by § XXXVII.) of the angle of the quaternion itself; if β' be (as in § LXIII.) the projection of β on α , then $S \cdot \beta\alpha = \beta'\alpha = \alpha\beta'$, and this scalar product (see again § XIII.) is positive or null or negative, according as the angle between α and β is obtuse, or right, or acute (contrast again § LXIV.); the projection β' may be expressed in terms of β and α , by writing $\beta' = \alpha^{-1} S \cdot \beta\alpha$, or $\beta' = a S \cdot \beta\alpha^{-1}$,

Articles 422 to 426; Pages 407 to 416.

§ LXVII. Vector of the product of two lines α , β ; if β'' denote (as in § LXIII.) the component of β which is perpendicular to α , then $V \cdot \beta\alpha = \beta''\alpha = a$ a line perpendicular to the plane of the two given factors α , β ; $V \cdot \beta\alpha \perp \alpha$, $V \cdot \beta\alpha \perp \beta$; the rotation round this vector of the product, from the multiplier line β , towards the multiplicand line α , is positive; whereas the positive rotation round the vector of the quotient $\beta \div \alpha$, or $\beta\alpha^{-1}$, is directed from α towards β ; $UV \cdot \beta\alpha = -UV \cdot \beta\alpha^{-1}$; the length of the vector of the product of two adjacent sides of a parallelogram represents the area of that parallelogram,

$$TV \cdot \beta\alpha = \sphericalangle \text{AOB} = T\beta T\alpha \sin \hat{\beta}\alpha;$$

$TVU \cdot \beta\alpha = \sin \hat{\beta}\alpha$ (compare § LXIII.); $V \cdot \alpha\beta = -V \cdot \beta\alpha$, the vector of the product of two lines changes sign (or direction) when the two factors are interchanged (whereas, by § LXV., $S \cdot \alpha\beta = +S \cdot \beta\alpha$); the perpendicular component β'' may be expressed in any one of the following ways,

$$\begin{aligned} \beta'' &= V \cdot \beta\alpha \div \alpha = -\alpha^{-1} V \cdot \beta\alpha = \alpha^{-1} V \cdot \alpha\beta \\ &= V \cdot \beta\alpha^{-1} \times \alpha = -\alpha V \cdot \beta\alpha^{-1} = \alpha V \cdot \alpha^{-1} \beta; \end{aligned}$$

new proof (compare § L.) that when $\gamma\alpha = a\beta$, then γ is the REFLEXION of the line β with respect to a ; the equation $V.\rho\alpha = V.\beta\alpha$, or $V.(\rho - \beta)\alpha = 0$, expresses that the termination P of ρ is situated on the *right line through B*, which is *parallel to a*, or to OA ; the same RECTILINEAR LOCUS of P may be expressed by writing $\rho = \beta + x\alpha$, where x denotes a *variable scalar*; the equation $V.\rho\alpha = 0$ denotes the indefinite right line *through the origin O*, of which the given line OA is a part; $V.\rho\alpha = V.a\beta$ denotes another indefinite right line, parallel to the line OA , and passing through a point c , which is the *reflexion* of the point B with respect to the line OA ; the equation $V(\rho V.\beta\alpha) = 0$, or $V.\rho V.\beta\alpha = 0$, expresses that ρ is *perpendicular to the plane AOB* of α and β ; whereas the equation $S.\rho V.\beta\alpha = 0$ (afterwards abridged, see § LXXXVI., to the form $S.\rho\beta\alpha = 0$), expresses that the three lines α, β, ρ , are *coplanar*, and gives therefore a PLANE as the locus of P; the equation,

$$(V.\rho\alpha)^2 = (V.\beta\alpha)^2, \text{ or } TV.\rho\alpha = TV.\beta\alpha,$$

denotes a CYLINDER OF REVOLUTION, with a for *axis*, and $T\beta'$ for *radius*; in like manner the equation $(V.\rho\beta^{-1})^2 + b^2 = 0$, or $TV.\rho\beta^{-1} = b$, represents another *cylinder of revolution*, with β for *axis*, and $bT\beta$ for *radius*,

Articles 427 to 431; Pages 416 to 423.

§ LXVIII. If we cut the last cylinder by the perpendicular plane $S.\rho\beta^{-1} = a$, the section is a CIRCLE, contained on the sphere $T\rho = (a^2 + b^2)^{\frac{1}{2}}T\beta$; the sphere round origin with radius $T\beta$, namely, the sphere for which $T\rho = T\beta$, or $T.\rho\beta^{-1} = 1$, may have its equation thus transformed, $(S.\rho\beta^{-1})^2 - (V.\rho\beta^{-1})^2 = 1$, and may be regarded as the *locus of a varying circle*, for which $S.\rho\beta^{-1} = x$, $TV.\rho\beta^{-1} = (1 - x^2)^{\frac{1}{2}}$; the first of these two equations of the circle represents here a *varying plane*, and the second represents a *varying cylinder of revolution*; if a be *inclined* to β , the cylinder $TV.\rho\beta^{-1} = b$ is cut *obliquely* by the plane $S.\rho\alpha^{-1} = a$ in an ELLIPSE; in like manner the equations, $S.\rho\alpha^{-1} = x$, $TV.\rho\beta^{-1} = (1 - x^2)^{\frac{1}{2}}$, represent a *varying ellipse*, of which the LOCUS (obtained by elimination of x) is an ELLIPSOID, represented by the equation,

$$(S.\rho\alpha^{-1})^2 - (V.\rho\beta^{-1})^2 = 1;$$

geometrical illustration of this mode of *generating an ellipsoid* by a certain *deformation of a sphere* (ellipses being substituted for circles, by substituting *oblique* for *perpendicular* sections of a certain *varying cylinder*); the ellipsoid is ENVELOPED by the cylinder of revolution, whose equation is $(V.\rho\beta^{-1})^2 = -1$; the plane of the *ellipse of contact* is $S.\rho\alpha^{-1} = 0$; the *equation of the ellipsoid* may also be thus written, $(S.\rho\alpha^{-1})^2 + (TV.\rho\beta^{-1})^2 = 1$; or thus, $T(S.\rho\alpha^{-1} + V.\rho\beta^{-1}) = 1$; this last form will be found to furnish (in §§ LXXVIII., &c.) a *new mode of generating the ellipsoid* (or rather a *number* of such new modes),

Articles 432 to 436; Pages 423 to 430.

§ LXIX. Analogous deformations of other surfaces of revolution; the locus of the *varying circle*, $S.\rho\beta^{-1} = x$, $TV.\rho\beta^{-1} = (x^2 - 1)^{\frac{1}{2}}$, is an EQUILATERAL

AND DOUBLE-SHEETED HYPERBOLOID OF REVOLUTION, whose equation is $(S \cdot \rho\beta^{-1})^2 + (V \cdot \rho\beta^{-1})^2 = 1$; the locus of the connected and varying *ellipse*, $S \cdot \rho\alpha^{-1} = x$, $TV \cdot \rho\beta^{-1} = (x^2 - 1)^{\frac{1}{2}}$, where α is still supposed to be inclined to β , is *another double-sheeted hyperboloid*, which is *not* one of revolution, and which has for its equation the following,

$$(S \cdot \rho\alpha^{-1})^2 + (V \cdot \rho\beta^{-1})^2 = 1;$$

geometrical illustrations: the right and oblique CONES, which are respectively ASYMPTOTIC to these two hyperboloids, have their equations formed by changing 1 to 0 in the second members of the equations of those two surfaces; by changing 1 to -1 in the same second members, we get the equations of two SINGLE-SHEETED HYPERBOLOIDS, with the *same* asymptotic cones, of which two hyperboloids the first is *equilateral* and of *revolution*, while the second *touches the ellipsoid* of § LXVIII. *along the ellipse of contact* mentioned in that section, namely, the ellipse whose equations are,

$$S \cdot \rho\alpha^{-1} = 0, \quad TV \cdot \rho\beta^{-1} = 1;$$

the second of the two double-sheeted hyperboloids touches the same ellipsoid at the extremities of the two opposite vectors which have the directions of $\pm \beta$, the *common tangent planes* at those two points being given by the formula $S \cdot \rho\alpha^{-1} = \pm 1$; the equations,

$$S \cdot \rho\beta^{-1} + (V \cdot \rho\beta^{-1})^2 = 0, \quad S \cdot \rho\alpha^{-1} + (V \cdot \rho\beta^{-1})^2 = 0,$$

represent two ELLIPTIC PARABOLOIDS, whereof the first is a surface of revolution; the equation $S \cdot \rho\alpha^{-1} S \cdot \rho\beta^{-1} = S \cdot \rho\gamma^{-1}$ represents an HYPERBOLIC PARABOLOID; an ARBITRARY SURFACE OF REVOLUTION may be represented by the formula, $TV \cdot \rho\beta^{-1} = f(S \cdot \rho\beta^{-1})$, and then the connected equation, $TV \cdot \rho\beta^{-1} = f(S \cdot \rho\alpha^{-1})$ will represent the result of a certain DEFORMATION of that surface, whereby ellipses are still substituted for circles; but if α be supposed to be *not inclined* to β , but only to be *longer or shorter*, the results of all the foregoing deformations will themselves be surfaces of revolution, . . . Articles 437 to 440; Pages 430 to 435.

§ LXX. MacCullagh's MODULAR GENERATION of surfaces of the second order, expressed in the language of quaternions; origin being on a *directrix*, α being vector of a *focus*, β vector of another point of *directrix*, and γ perpendicular to a *directive plane*, the following equation may be established, $T(\rho - \alpha) = T(\rho S \cdot \gamma\beta - \beta S \cdot \gamma\rho)$; it will be found (see § XCL.) that this equation admits of being put under the form

$$T(\rho - \alpha) = TV \cdot \gamma V \cdot \beta\rho, \quad . . .$$

Article 441; Pages 435 to 437.

§ LXXI. The symbol $V(V \cdot \alpha\beta \cdot V \cdot \gamma\delta)$ denotes a *line* situated in the *intersection of the two planes* of α, β , and of γ, δ ; if there be *six* diverging vectors $\alpha, \alpha', \dots, \alpha^v$, and if we form from them *three* others, β, β', β'' , by the formulæ,

$$\begin{aligned}\beta &= V(V \cdot \alpha \alpha' \cdot V \cdot \alpha'' \alpha^{iv}), \\ \beta' &= V(V \cdot \alpha' \alpha'' \cdot V \cdot \alpha^{iv} \alpha^v), \\ \beta'' &= V(V \cdot \alpha'' \alpha''' \cdot V \cdot \alpha^v \alpha),\end{aligned}$$

then the equation, $0 = S \cdot \beta \beta' \beta''$, expresses the condition for the six diverging lines, $\alpha, \alpha', \dots \alpha^v$, being *six sides of one common cone* of the second degree, and may therefore be called the EQUATION OF HOMOCONICISM; the scalar function $S \cdot \beta \beta' \beta''$ may be called the ACONIC FUNCTION of the six vectors $\alpha \dots \alpha^v$, or of the HEXAGON (plane or gauche) at whose corners they terminate, because it *vanishes* when they are *homoconic*, by a form of the theorem of Pascal; hence may be derived an expression by quaternions, for what may be called the ADEUTERIC FUNCTION OF TEN VECTORS, $\alpha, \alpha', \dots \alpha^{ix}$, or of the (generally gauche) DECAGON at whose corners they terminate, because this function *vanishes, when those TEN POINTS are on one COMMON DEUTERIC SURFACE, or common surface of the second order*; the *Adeuteric* may be thus expressed,

$$\Sigma (\pm ABCDEF. GHJK),$$

if $\Lambda \dots \kappa$ be the ten points, while the symbol ABCDEF here denotes the *aconic* function of six of them, with respect to any eleventh point o arbitrarily taken as an origin, and GHJK denotes the *pyramidal* function of the other four, that is, the *sextupled volume of the pyramid* of which they are the corners, taken with a proper algebraic sign; in symbols, this pyramidal function of four points, $\alpha, \beta, \gamma, \delta$, or of four vectors, $\alpha^{vi}, \alpha^{vii}, \alpha^{viii}, \alpha^{ix}$ may be expressed by quaternions as follows:

$$S \cdot (\alpha^{ix} - \alpha^{vi}) (\alpha^{viii} - \alpha^{vi}) (\alpha^{vii} - \alpha^{vi}) \text{ (compare § LXXXIX.)};$$

the ten points are supposed to be combined in all possible ways, as groups of four and six (namely in 210 ways), by successive mutual interchanges of points or of letters between the two groups; for every such binary interchange the sign \pm prefixed to the product varies; this formation of the adeuteric function is only alluded to in the text of the Lecture, . . .

Article 442; Pages 437 to 439.

§ LXXII. The general *addition of any two quaternions* can always be easily and *definitely* effected by the *rule of the common operand*, or by the formula $(\gamma \div \alpha) + (\beta \div \alpha) = (\gamma + \beta) \div \alpha$; *subtraction* of quaternions may in like manner be effected by the formula $(\gamma \div \alpha) - (\beta \div \alpha) = (\gamma - \beta) \div \alpha$;

Articles 443 to 447; Pages 439 to 444.

§ LXXIII. Properties of such addition; it is a *commutative* and *associative* operation; the scalar, vector, and conjugate of a *sum* of quaternions are respectively the sums of the scalars, vectors, and conjugates, $S\Sigma = \Sigma S$, $V\Sigma = \Sigma V$, $K\Sigma = \Sigma K$; similarly for *differences*, $S\Delta = \Delta S$, $V\Delta = \Delta V$, $K\Delta = \Delta K$; it is useful to be familiar with the two following general expressions, for the scalar and vector parts of the product of any two vectors, $S \cdot \alpha\beta = \frac{1}{2}(\alpha\beta + \beta\alpha)$, $V \cdot \alpha\beta = \frac{1}{2}(\alpha\beta - \beta\alpha)$, Articles 448, 449; Pages 444 to 447.

§ LXXIV. The *general QUADRINOMIAL FORM*, $q = w + ix + jy + kz$, for a quater-

nion, may now be more fully understood; $q' = w' + ix' + jy' + kz'$ being another quadrinomial of the same sort, the sum and difference of these two quaternions are formed by taking the sums and differences of their CONSTITUENTS, w, x, y, z and w', x', y', z' ; in symbols, $q' \pm q = w' \pm w + i(x' \pm x) + j(y' \pm y) + k(z' \pm z)$; a quaternion cannot *vanish*, except by its four constituents *separately* vanishing; nor can two quaternions become *equal*, without their constituents becoming *separately* equal; an equation $q' = q$ between two quaternions includes thus a SYSTEM OF FOUR EQUATIONS between scalars; namely, $w' = w, x' = x, y' = y, z' = z, \dots$

Article 450; Pages 447 to 449.

§ LXXV. General proof of the DISTRIBUTIVE PRINCIPLE of multiplication of quaternions; $\Sigma r . \Sigma q = \Sigma . r q$; . . . Articles 451 to 455; Pages 449 to 455.

§ LXXVI. Elementary applications of the distributive principle; transformations by means of it, referred to in § LXIV.; the equation or identity,

$$(\alpha - \beta)^2 = \alpha^2 - 2S . \alpha\beta + \beta^2,$$

is equivalent to the *fundamental formula of plane trigonometry*, or to the equation,

$$\overline{BA}^2 = \overline{CA}^2 - 2\overline{CA} . \overline{CB} . \cos \angle ACB + \overline{CB}^2;$$

centre of mean distances, or of gravity, $\mu = \Sigma . aa \div \Sigma a$; investigation of the (spherical) locus of the vertex of a triangle, of which the base and the ratio of the sides are given; $T(\sigma - n\gamma) = T(n\sigma - \gamma)$, if $T\sigma = T\gamma, \dots$

Articles 456 to 459; Pages 455 to 460.

§ LXXVII. Intersections of right line and sphere; the locus of all the tangents to the sphere $\rho^2 + c^2 = 0$, which can be drawn from the extremity of β , has for equation, $c^2(\rho - \beta)^2 = (V . \beta\rho)^2$; this form of the equation of the enveloping cone is simpler than that which was obtained in § LXVI., but the one can be transformed into the other; new investigation of the equation of the polar plane, $S . \beta\rho = -c^2$ (compare again § LXVI.); proof by quaternions, of the known harmonic property of this plane; HARMONIC MEAN BETWEEN ANY TWO VECTORS; fourth harmonical to any three points (not necessarily on one straight line); extension hereby given to the usual notion of harmonic conjugates; circular harmonic group (four points on a circle, for which what is called the anharmonic quotient becomes unity); interpretations of the sum and difference of the reciprocals of any two vectors, Articles 460 to 464; Pages 460 to 466.

§ LXXVIII. Equation of ellipsoid resumed (from § LXVIII.), and transformed to

$$T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2;$$

geometrical equality hence deduced,

$$\overline{AE} = \overline{BD};$$

GENERATION OF THE ELLIPSOID, hence derived; if A be a superficial point of a fixed sphere with centre c, and B an external point, and if a secant BDD' be drawn, and on the guide-chord AD, or on that chord either way

prolonged, a portion ΔE be taken, which in *length* is equal to BD' , the I.O.CUS of the point E will be an ellipsoid, with A for its centre, and B for a point of its surface; ABC in this construction may be called the GENERATING TRIANGLE, and the sphere round C the DIACENTRIC SPHERE; the points D and D' on that sphere may be said to be conjugate guide-points; geometrical deductions from the formula, $\overline{AE} = \overline{BD}'$; constructions for the lengths and directions of the three principal semi-axes of the ellipsoid, a , b , c ; expressions for the lengths of the sides of the generating triangle,

$$\overline{BC} = \frac{1}{2}(a+c), \overline{CA} = \frac{1}{2}(a-c), \overline{AB} = acb^{-1};$$

enveloping cylinder of revolution, with the side AB for axis, and $BG = b$ for radius, if G be the second point of intersection of AB with the diacentric sphere; the two other sides, BC , CA , of the triangle are perpendicular to the two cyclic planes of the ellipsoid; the one that is $\perp \kappa$, or $\perp CA$, touches the diacentric sphere at A ; these planes are also shewn by this construction to be (as is known) the cyclic planes of all the concentric cones, that rest on those SPHERICAL CONICS in which the ellipsoid is cut by a system of concentric spheres; MEAN SPHERE, containing the two diametral and circular sections; the construction exhibits also geometrically the known mutual rectangularity of the semi-axes ΔE_1 , ΔE_2 of any other diametral section of the ellipsoid, and conducts easily to the known expression for the difference of the squares of their reciprocals, namely,

$$\overline{\Delta E_2}^2 - \overline{\Delta E_1}^2 = (c^2 - a^2) \sin v \sin v',$$

where v and v' are the inclinations of the cutting plane to the two cyclic planes; the equations of these latter planes are, respectively, $S \cdot \iota \rho = 0$, $S \cdot \kappa \rho = 0$; the equation of the mean sphere is

$$T\rho = b = (\kappa^2 - \iota^2)T(\iota - \kappa)^{-1};$$

$$a = T\iota + T\kappa, c = T\iota - T\kappa, ac = \kappa^2 - \iota^2, acb^{-1} = T(\iota - \kappa);$$

equations of a spherical conic on the ellipsoid; expressions for the two new vectors, ι , κ , as functions of the vectors, α , β , of § LXVIII.,

Articles 465 to 470; Pages 466 to 475.

§ LXXIX. Introduction of two new vectors, λ , μ , with two new scalars, h , h' , and two new points, L , M , which all depend upon and vary with the vector ρ , or the point E , and satisfy the equations,

$$\lambda = (\kappa\rho + \rho\kappa)(\kappa - \iota)^{-1} = h(\iota - \kappa) = \Delta L = h \cdot \Delta B,$$

$$\mu = (\iota\rho + \rho\iota)(\iota - \kappa)^{-1} = h'(\kappa - \iota) = \Delta M = h' \cdot \Delta A;$$

to each given value of h (between certain limits) answers a circle on the ellipsoid, for which

$$S \cdot \kappa \rho = \frac{1}{2} h T(\iota - \kappa)^2, \overline{LE} = T(\rho - \lambda) = b;$$

in like manner, to each given value of h' (suitably limited) there answers another circle on the ellipsoid, determined by the equations,

$$S \cdot \iota \rho = \frac{1}{2} h' T(\iota - \kappa)^2, \overline{ME} = T(\rho - \mu) = b;$$

these two *subcontrary* and circular sections of the ellipsoid have their planes *perpendicular* to the sides, CA, CB of the generating triangle (§ LXXVIII.), and therefore *parallel* (as is known) to the two cyclic planes; every such *pair* of subcontrary circles (*h, h'*) is contained (as by known results it ought to be) on *one common sphere*; this sphere, in these calculations, is given by the formula,

$$T(\rho - \xi) = \overline{NE} = n,$$

where the vector ξ , the positive scalar n , and the point N , may be determined by the equations,

$$AN = \xi = hu + h'\kappa, b^2 - n^2 = (h + h')(hu^2 + h'\kappa^2);$$

and if we make $EN = \xi - \rho = b^2\nu$, then N is the *foot of the normal* to the ellipsoid drawn at the point E , and terminated by the plane of the generating triangle, or by the plane of the greatest and least axes, while n denotes the *length* of that normal; the new vector ν is parallel to the normal, and satisfies the equation $S \cdot \nu\rho = 1$; its expression as a function of ρ is,

$$\nu = (\kappa^2 - \iota^2)^{-2} \{ (\iota - \kappa)^2 \rho + 2\iota \cdot S \cdot \kappa\rho + 2\kappa S \cdot \iota\rho \};$$

the equation of the ellipsoid may be put under the form, $\rho^2 + b^2 = \lambda\mu$, while that of the mean sphere may be thus written, $\rho^2 + b^2 = 0$, . . .

Articles 471 to 474; Pages 476 to 479.

§ LXXX. If we make for abridgment $\nu = \phi(\rho)$, or simply $\nu = \phi\rho$, the *vector function* ϕ will be *linear* or *distributive*,

$$\phi(\rho + \rho') = \phi\rho + \phi\rho', \Delta\phi\rho = \phi\Delta\rho, \phi(x\rho) = x\phi\rho;$$

and if we agree to write $f(\rho, \varpi) = S \cdot \rho\phi\varpi$, the *scalar function* f will be at once *commutative* or *symmetric* with respect to the two vectors on which it depends, and *linear* or *distributive* relatively to *each* of them, so that $f(\varpi, \rho) = f(\rho, \varpi)$, $f(\rho + \rho', \varpi + \varpi') = f(\rho, \varpi) + f(\rho, \varpi') + f(\rho', \varpi) + f(\rho', \varpi')$, $f(x\rho, y\varpi) = xyf(\rho, \varpi)$; if then we farther abridge $f(\rho, \rho)$ to $f(\rho)$ or to $f\rho$, this *new* scalar function of *one* vector will, relatively to *it*, be of the *second* dimension, and we shall have

$$f(\rho + \rho') = f\rho + 2f(\rho, \rho') + f\rho', f(x\rho) = x^2f\rho;$$

the *equation of the ellipsoid* reduces itself in this notation to the formula, $f\rho = 1$; and if a *cylinder* (not generally of revolution) be *circumscribed* about the ellipsoid, with its generating lines parallel to a given vector ϖ , the equation $f(\rho, \varpi) = 0$ represents the *diametral plane of contact*, and the *normal* to that plane has the direction of the vector $\phi\varpi$; in general the last equation denotes that the *directions* of ρ and ϖ are *conjugate*, relatively to the ellipsoid; reciprocal relations of bisection, conjugation of line and plane, system of three conjugate semi-diameters, equation $x^2 + y^2 + z^2 = 1$, Articles 475 to 480; Pages 480 to 485.

§ LXXXI. The equation $f(\rho, \varpi) = 1$, or $S \cdot \nu\varpi = 1$, expresses that the vector ϖ terminates on the *tangent plane* to the ellipsoid, drawn at the extremity of the

semi-diameter ρ ; the vector ν , or $\phi\rho$, may be called the VECTOR OF PROXIMITY, namely, of the tangent plane to the centre, because its reciprocal ν^{-1} represents in length and in direction the perpendicular let fall from that centre on that plane; in general the formula $f(\rho, \varpi) = 1$ may be said to be the equation of conjugation between the two vectors ρ and ϖ , because it expresses that they terminate in two conjugate points; the same equation represents the polar plane of either of those two points, when the other is treated as variable; if ϖ be treated as the vector of the vertex of an enveloping cone, the equation of that cone is

$$\{f(\rho, \varpi) - 1\}^2 = (f\rho - 1)(f\varpi - 1):$$

when the vertex goes off to infinity, there results an enveloping cylinder, with the equation $f(\rho, \varpi)^2 = (f\rho - 1)f\varpi$; verifications for the case of the sphere, for which $\kappa = 0$, $\phi\rho = \iota^{-2}\rho$; general harmonic property of the polar plane, Articles 481 to 486; Pages 485 to 491.

§ LXXXII. The triangles LMN, ABC, are similar and similarly situated in one common plane; the points B, D, E, L are concircular; the triangle LEM is isosceles; the lines LN, MN are portions of the axes of the two circles on the ellipsoid which pass through the point E, Articles 487, 488; Pages 491, 492.

§ LXXXIII. New proof of the associative principle of multiplication of quaternions, derived from the distributive principle; importance of combining these two principles, Articles 489, 490; Pages 493 to 495.

§ LXXXIV. Transformed equation of the ellipsoid,

$$T(\iota\rho + \rho\kappa') = \kappa'^2 - \iota'^2; \iota\kappa' = \iota\kappa = T.\iota\kappa;$$

new generating triangle $\Lambda B'C'$, and new diacentric sphere round c' , touching at A the cyclic plane $\perp \iota$ (compare § LXXVIII.); $\Lambda B'$ is the axis of a second enveloping cylinder of revolution; if we make (compare § LXXIX.),

$$\Lambda L' = \lambda' = 2(\kappa' - \iota')^{-1} S.\kappa'\rho, \Lambda M' = \mu' = 2(\iota' - \kappa')^{-1} S.\iota'\rho,$$

the two new triangles, $\Lambda'M'N$ and $\Lambda B'C'$ are similar and similarly situated in one common plane, namely, in the principal plane of the ellipsoid; the symbols $V^{-1}0$, $S^{-1}0$, denote respectively a scalar and a vector; when three points are collinear, the vector part of the quotient of the differences vanishes and conversely; $LMM'L$ is a quadrilateral in a circle, whereof the diagonals LM' , ML' intersect in N , that is (§ LXXIX.), in the foot of the normal to the ellipsoid; GENERATION OF A SYSTEM OF TWO RECIPROCAL ELLIPSOIDS, by means of a MOVING SPHERE; generation of the same system of two ellipsoids by means of a FIXED SPHERE; if the sides of a plane quadrilateral inscribed in the fixed sphere move parallel to four fixed lines, one pair of opposite sides will intersect in a point on one ellipsoid, and the other pair of opposite sides will intersect in the corresponding point on the other or reciprocal ellipsoid; these two ellipsoids have one common mean sphere, namely, the fixed sphere employed in the construction; other geometrical relations of the fixed sphere and lines to the two ellipsoids thus generated, Articles 491 to 495; Pages 495 to 502.

§ LXXXV. Generation of an ellipsoid by means of a PAIR OF SLIDING SPHERES ; if two equal spheres slide within two cylinders of revolution, whose axes intersect each other, in such a manner that the right line joining their centres moves parallel to a fixed line, the *locus of their circle of intersection is an ellipsoid*, inscribed at once in both the cylinders; the same ellipsoid may also be generated as the locus of the circular intersection of *another pair of sliding spheres*, inscribed within the same two cylinders, but with their line of centres parallel to a different straight line; the diameter of each sliding sphere is equal to the mean axis $2b$ of the ellipsoid; an *arbitrary curve on the surface of the ellipsoid may be described by the vertex E of an isosceles triangle LEM' (or L'EM)*, the common length of whose two sides EL, EM' (or EL', EM) is constant, and $= b$, while its base LM' (or L'M) moves parallel to a given line AC (or AC'), and is inscribed in a given angle BAB'; or a *rhombus of constant perimeter, = 4b*, may be employed to generate, in an analogous way, by the motions of two opposite corners, *two curves* on the ellipsoid, Article 496; Pages 502, 503.

§ LXXXVI. Introduction of two new fixed vectors, $\eta = T\iota U (\iota - \kappa)$, $\theta = T\kappa U (\iota' - \kappa')$; making $g = -h' T(1 - \kappa\iota^{-1})$, we have $\mu = g\eta$, $\lambda' = g\theta$, and the equations of one pair of sliding spheres become

$$T(\rho - g\eta) = T(\rho - g\theta) = b;$$

for any one value of the variable scalar g , the plane of the circle of intersection is represented by the equation,

$$g(\theta^2 - \eta^2) = 2S.(\theta - \eta)\rho,$$

and we have the value, $\eta - \theta = b U\iota$; elimination of g gives for the ellipsoid, regarded as the locus of these circles, the transformed equation,

$$TV \frac{\eta\rho - \rho\theta}{U(\eta - \theta)} = \theta^2 - \eta^2, \text{ or, } TV \frac{\eta\rho - \rho\theta}{\eta - \theta} = \frac{\theta^2 - \eta^2}{T(\eta - \theta)};$$

other mode of obtaining this last equation from the form in § LXXXVIII., namely, $T(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2$; in general, for any three vectors α, β, γ , we have the identities,

$$S. \alpha\beta\gamma = -S. \gamma\beta\alpha, V. \alpha\beta\gamma = +V. \gamma\beta\alpha,$$

with analogous results (compare §§ LIIT., LXIII.) for the *scalar and vector of the product of any odd number of vectors*; we have also, generally,

$$S. \gamma V. \beta\alpha = S. \gamma\beta\alpha, S. \gamma Vq = S. \gamma q;$$

a *fraction* in this calculus may generally be transformed (as in Algebra), by *dividing both* numerator and denominator *by* any common vector or quaternion distinct from zero; or, in other words, by *multiplying each into* (but *not generally by*) the *reciprocal* of any such vector or quaternion, .

Articles 497 to 500; Pages 503 to 509.

§ LXXXVII. Geometrical significations of the two new fixed vectors, η, θ ; $\eta + \theta = \omega$ is the vector of an UMBILIC of the ellipsoid, and the equation of the

tangent plane at that umbilic (found by making $g=2$) is $S \cdot (\theta - \eta) \rho = \theta^2 - \eta^2$; the *umbilical normal* there has the direction of $\eta - \theta$, or of the cyclic normal ι ; $\theta^{-1} - \eta^{-1}$ has the direction of the *other* cyclic normal κ ;

$$\begin{aligned} \iota &= T\eta U (\eta - \theta), \quad \kappa = T\theta U (\theta^{-1} - \eta^{-1}); \\ a &= T\eta + T\theta, \quad b = T (\eta - \theta), \quad c = T\eta - T\theta; \end{aligned}$$

the sum and difference $U\eta \pm U\theta$ are respectively equal to $U (\iota - \kappa) \pm U (\iota' - \kappa')$, and have the directions of the greatest and least axes of the ellipsoid; the *length* of an umbilical vector, or *umbilical semi-diameter* of the ellipsoid, is

$$u = T\omega = T (\eta + \theta) = \sqrt{a^2 - b^2 + c^2};$$

the length of the *perpendicular* from the centre on the umbilical tangent plane is

$$p = (\theta^2 - \eta^2) T (\eta - \theta)^{-1} = acb^{-1};$$

these values of u and p agree with known results; *another* umbilical vector is

$$\omega' = T\eta U\theta + T\theta U\eta = -T \cdot \eta\theta \cdot (\eta^{-1} + \theta^{-1});$$

$-\omega, -\omega'$ are also umbilical vectors; thus $\eta^{-1} + \theta^{-1}$ has the direction of such a vector;

$$\begin{aligned} \omega + \omega' &= (T\eta + T\theta) (U\eta + U\theta), \\ \omega - \omega' &= (T\eta - T\theta) (U\eta - U\theta), \end{aligned}$$

the angles between the umbilical diameters are seen to be bisected by the greatest and least axes, Articles 501 to 503; Pages 509 to 511.

§ LXXXVIII. For the *square* of any quaternion we have the following scalar, vector, and tensor,

$$S \cdot q^2 = Sq^2 + Vq^2, \quad V \cdot q^2 = 2VqSq, \quad T \cdot q^2 = Sq^2 - Vq^2;$$

hence for the *scalar of the square root* of any other quaternion q' we have the expression,

$$S \sqrt{q'} = \sqrt{\frac{1}{2}Sq' + \frac{1}{2}Tq'};$$

this is only *one* out of a vast number of *general transformations*, in which the present CALCULUS abounds, and which may be deduced from the *laws of the symbols* S, T, U, V, K; applied to the ellipsoid, in combination with the recent values for a, b, c , it enables us to infer that the linear *eccentricities* of the two sections, perpendicular respectively to the mean and greatest axes, are,

$$(a^2 - c^2)^{\frac{1}{2}} = 2T \sqrt{(\eta\theta)}, \quad (b^2 - c^2)^{\frac{1}{2}} = 2S \sqrt{(\eta\theta)};$$

if we change at once θ to $t\theta$ and η to $t^{-1}\eta$, where t is any positive scalar, we pass to a CONFOCAL ELLIPSOID, the FOCAL ELLIPSE and FOCAL HYPERBOLA remaining still unchanged; the focal *ellipse* may conveniently be represented by the system of the two equations

$$S \cdot \rho U\eta = S \cdot \rho U\theta, \quad TV \cdot \rho U\eta = 2S \sqrt{(\eta\theta)},$$

which represent separately the plane of the ellipse, and a cylinder of revo-

lution on which the ellipse is contained; or we may combine the same plane with this other cylinder of revolution,

$$TV \cdot \rho U \theta = 2S \vee (\eta \theta);$$

the focal *hyperbola* is *adequately* represented, as a curve in *space*, by the *single equation*,

$$V \cdot \eta \rho \cdot V \cdot \rho \theta = (V \cdot \eta \theta)^2;$$

because this equation will be found to *include* within itself *the equation of the plane* of the hyperbola, namely, $S \cdot \rho \eta \theta = 0$, as well as the constancy of the *product of the projections* on the asymptotes, which asymptotes are here the lines η, θ , or (as is known) the axes of all the cylinders of revolution circumscribed about the ellipsoid and its confocals;

Articles 504, 505; Pages 511 to 513.

§ LXXXIX. In general, in this Calculus, a *scalar equation*, $f\rho = c$, involving one variable vector ρ , represents a *surface*; in fact it is *equivalent* to an *ordinary algebraic equation* between the *three* Cartesian co-ordinates x, y, z , and may be changed to such an equation by substituting for ρ its trinomial value $i x + j y + k z$ (see § XIX.); examples; the actual process of squaring the last-mentioned trinomial gives $\rho^2 = -x^2 - y^2 - z^2$; if we make $a = i a + j b + k c$, $a' = i a' + j b' + k c'$, then actual multiplication gives expressions for the products $a\rho, a'\rho$, of which the scalar parts are, respectively, $S \cdot a\rho = -(ax + by + cz)$, and $S \cdot a'\rho$ = the DETERMINANT

$$\begin{vmatrix} a, & b, & c, \\ a', & b', & c', \\ x, & y, & z; \end{vmatrix}$$

$$\text{or} = a(bz - cy) + b(cx - az) + c(ay - bx);$$

we have the two identities,

$$\begin{aligned} \rho S \cdot \gamma \beta \alpha &= \gamma S \cdot \rho \beta \alpha + \beta S \cdot \gamma \rho \alpha + \alpha S \cdot \gamma \beta \rho, \\ \rho S \cdot \gamma \beta \alpha &= V \cdot \beta \alpha S \cdot \gamma \rho + V \cdot \alpha \gamma S \cdot \beta \rho + V \cdot \gamma \beta S \cdot \alpha \rho, \end{aligned}$$

of which the second shews that the elimination of ρ between the three equations $S \cdot a\rho = 0, S \cdot \beta\rho = 0, S \cdot \gamma\rho = 0$, conducts to the equation $S \cdot \gamma \beta \alpha = 0$; *co-ordinates and quaternions* may thus be employed to assist and *illustrate each other*; additional examples; the symbol $S \cdot \gamma \beta \alpha$ denotes the *volume of the parallelepipedon* of which $a\beta\gamma$ are edges, this volume being taken *positively* or *negatively*, according as the rotation round γ from β to α is *negative* or *positive* (compare § XXXIX.); we might in this way see (compare § LXXXVI.) that this function $S \cdot \gamma \beta \alpha$ *changes sign*, when *any two* of its factors are interchanged; the *scalar of a product* does not alter, when its factors are *CYCLICALLY PERMUTED*, $S \cdot \gamma \beta \alpha = S \cdot \beta \alpha \gamma, S \cdot s r q = S \cdot r q s$, &c.,

Articles 506 to 512; Pages 513 to 521.

§ XC. An equation of *vector form*, $\phi\rho = \lambda$, where ϕ denotes a *vector function*, and λ a given vector, may in general be *resolved* into *three scalar equations*, which suffice (theoretically speaking) to determine generally x, y, z ,

and therefore also ρ , or at least to restrict those co-ordinates, and this vector, to a *finite variety* of values; examples; if q be a given quaternion, the equation $V. q\rho = \lambda$ gives $\rho Sq = \lambda + q^{-1}V. \lambda Vq$; *notations* $\frac{V}{S}$, &c.; other form for the solution of the last equation in ρ ; the equation $V. \beta\rho\gamma = \lambda$ gives $\rho = \frac{\beta\lambda\beta^{-1} + \gamma\lambda\gamma^{-1}}{\beta\gamma + \gamma\beta}$; interpretation of this expression, in connexion with the results of § XLII.; the sine of the semisum of the angles of the spherical triangle DEF is equal to the cosine of the common bisector AB of two sides, divided by the cosine of CD, namely, of the half of the third side; for any three vectors, we have the following transformation, which is very often useful in this calculus,

$$V. \beta\rho\gamma = \beta S. \gamma\rho - \rho S. \beta\gamma + \gamma S. \beta\rho, \dots$$

Articles 513 to 518; Pages 521 to 526.

§ xci. Other mode of deducing this general and useful equation of transformation; if Π' be used as the characteristic of the operation of taking a *product*, with an *inverted order of the factors*, then (by §§ LIII., LXIII.),

$$K\Pi = \Pi'K, S = \frac{1}{2}(1 + K), V = \frac{1}{2}(1 - K);$$

hence

$$S\Pi = \frac{1}{2}\Pi + \frac{1}{2}\Pi'K, V\Pi = \frac{1}{2}\Pi - \frac{1}{2}\Pi'K;$$

thus, whatever vectors a, β, γ, δ , may be, we have

$$S. \gamma\beta a = \frac{1}{2}(\gamma\beta a - a\beta\gamma), V. \gamma\beta a = \frac{1}{2}(\gamma\beta a + a\beta\gamma);$$

$$S. \delta\gamma\beta a = \frac{1}{2}(\delta\gamma\beta a + a\beta\gamma\delta), V. \delta\gamma\beta a = \frac{1}{2}(\delta\gamma\beta a - a\beta\gamma\delta), \&c.;$$

and the identity, $\frac{1}{2}(\gamma\beta a + a\beta\gamma) = \frac{1}{2}\gamma(\beta a + a\beta) - \frac{1}{2}(\gamma a + a\gamma)\beta + \frac{1}{2}a(\gamma\beta + \beta\gamma)$, gives $V. \gamma\beta a = \gamma S. \beta a - \beta S. \gamma a + a S. \beta\gamma$, a result agreeing with the last section; we have also (compare § LXX.), these two other formulæ of transformation,

$$V. \gamma V. \beta a = a S. \beta\gamma - \beta S. a\gamma; V(V. \gamma\beta. a) = \gamma S. \beta a - \beta S. a\gamma;$$

the student ought to make himself very familiar with the three last formulae, which are valid for any three vectors; we have also, for any four vectors,

$$S. a'' a' a' a = S. a'' a S. a' a'' - S. a'' a' S. a'' a + S. a'' a' S. a a';$$

$$S(V. a'' a'' . V. a' a) = S. a'' a . S. a' a'' - S. a'' a' . S. a' a;$$

the comparison of the two expressions for $V(V. a'' a'' . V. a' a)$ conducts to the first identity of § LXXXIX.; as included in which, it is shewn that if a, a' be two non-parallel vectors, and $a'' = V. a' a$, then an arbitrary vector ρ may be expressed as follows,

$$\rho = a S \frac{a' \rho}{a''} + a' S \frac{\rho a}{a''} + \frac{S. a'' \rho}{a''}, \dots$$

Articles 519 to 523; Pages 526 to 529.

§ XCII. Connexion of quaternions with *spherical trigonometry*; the expression recently given for the scalar part of the product of the vector parts of two binary products of vectors may be interpreted as equivalent to the following theorem of Gauss,

$$\cos LL'. \cos L'L'' - \cos LL''. \cos L'L' = \sin LL'. \sin L''L' \cos A,$$

where A is the spherical angle between the arcs $LL', L''L''$; there are various ways of deducing from quaternions the fundamental formula, $\cos b = \cos c \cos a + \sin c \sin a \cos B$; if the rotation round β from a towards γ be positive,

$$V. \gamma\beta . V. \beta\alpha = \sin a \sin c (\cos + \beta \sin) B;$$

$$\tan \alpha \beta \gamma = \tan B = \beta^{-1} \frac{V}{S} (V. \gamma\beta . V. \beta\alpha), \dots$$

Articles 524 to 526; Pages 529 to 532.

§ XCIII. Connexion of quaternions with *goniometry*, or with the doctrine of *functions of angles*; a and ι being any two unit-vectors, and t any scalar, we have $S. a^t = S. \iota^t = f(t) = ft = a$ scalar and *even* function of t ; $a^t = ft + af(t-1)$, $\iota^t = ft + \iota f(t-1)$; $f(-t) = ft$, $f(2 \mp t) = -ft$; $f(u+t) = fuft - f(u-1)f(t-1)$; $(ft)^2 + \{f(t-1)\}^2 = 1$; $f(\frac{1}{2}t) = (\frac{1}{2} + \frac{1}{2}ft)\frac{1}{2}$; the values of ft may be *numerically calculated* and tabulated; the function f of a *multiple* of t may be transformed by the help of the equation,

$$2f(nt) = \{ft + \iota f(t-1)\}^n + \{ft - \iota f(t-1)\}^n;$$

the consideration of a *small rotation* gives the *differential expression*, $d. \iota^t = \frac{\pi}{2} \iota^{t+1} dt$; hence $f^t = \frac{\pi}{2} f(t+1)$, $f^t + \left(\frac{\pi}{2}\right)^2 ft = 0$; $f0 = 1$, $f'0 = 0$; *developements* for ft and $f(t-1)$; $\iota^t = e^{\frac{1}{2}\pi t}$, this exponential symbol being here employed merely as a *concise expression for a series* of well-known form; with the usual notations for cosine and sine, $ft = \cos \frac{\pi t}{2}$, $\iota^t = \cos \frac{\pi t}{2} + \iota \sin \frac{\pi t}{2}$; the equation $\gamma^z \beta \nu a^x = -1$, of § XLIX., under the form $\gamma^{2-z} = \beta \nu a^x$, may be expanded into the following, $\cos(\pi - C) + \gamma \sin(\pi - C) = (\cos B + \beta \sin B)(\cos A + a \sin A)$; the comparison of *scalars* gives a known and fundamental formula of spherical trigonometry, from which all others might be deduced, namely, $-\cos C = \cos B \cos A - \cos c \sin B \sin A$; the comparison of *vectors* gives

$$\gamma \sin C = a \sin A \cos B + \beta \sin B \cos A + V. \beta a . \sin A \sin B,$$

which may be interpreted as a theorem respecting the construction of a parallelepipedon, connected with a spherical triangle; *addition* of quaternions, and the *distributive* character of their multiplication, might be illustrated by spherical trigonometry, . . . Articles 527 to 529; Pages 532 to 537.

§ XCIV. Brief account of some early investigations by the present writer, whereby he was led (in 1843) to results agreeing in substance with those lately mentioned, respecting the connexions of quaternions with spherical trigo-

nometry; *symbolic multiplication table*, for the squares and products of i, j, k ; developement of a *product* of two quaternions, under their quadri-nomial forms; reproduction of a theorem of Euler, respecting the products of *sums of four squares*; subsequent extension (in the same year) by J. T. Graves, Esq., to a theorem respecting *sums of eight squares*, and to a theory of certain *octaves*, involving *seven* distinct imaginaries; allusion to subsequent publications of Professor De Morgan, and other mathematicians of these countries, in the same general field of research, or at least on ana-logous subjects, such as the *triplets*, *tessarines*, and *pluquaternions*; the writer regrets that it is not possible for him here to analyze, or even to enumerate, those important and interesting publications; the quaternions early conducted him to a general theorem respecting *spherical polygons*, which includes as a particular case the following theorem respecting a spherical triangle, and may in turn be derived from it,

$$(\cos C + \gamma \sin C) (\cos B + \beta \sin B) (\cos A + \alpha \sin A) = -1;$$

this particular theorem may be expressed by the lately cited formula of § XLIX., $\gamma^2\beta^2\alpha^2 = -1$; the more general theorem for a polygon may be expressed by an analogous equation, namely, $\alpha_n^{\alpha_{n-1}} \dots \alpha_1^{\alpha_1} \alpha^\alpha = (-1)^n$; another early and general theorem of this calculus, respecting spherical polygons, which is a sort of *polar transformation* of the foregoing, may be expressed by a connected formula, . Articles 530 to 536; Pages 537 to 545.

§ XCV. *Exponential Functions*, direct and inverse; the *tensor of the sum* of any number of quaternions cannot exceed the sum of the tensors; if we write

$$F_m q = 1 + \frac{q}{1} + \frac{q^2}{1.2} + \dots + \frac{q^m}{1.2\dots m},$$

the number m may be assumed *so large*, however large the *given tensor* of the quaternion q may be, that the *last term* (reading *here* from left to right) may have its *tensor less* than any *given* and *positive* quantity, b ; and not only so, but that the *quaternion sum* of the n following terms of the same series, or the *quaternion difference* $F_{m+n}(q) - F_m(q)$, shall also have its tensor $< b$, however large the number n of these new terms may be; the finite series $F_m q$ converges to a definite quaternion limit, $F_\infty q$ or Fq , when the number m of terms increases indefinitely; the resulting function, Fq , has the well-known EXPONENTIAL CHARACTER, whenever the condition of commutativeness is satisfied; $Fr \cdot Fq = F(r + q)$ if $r q = q r$; for example, we have, generally, $Fq = FSq \cdot FVq$, where it is found that FSq is a positive scalar, and FVq is a versor, so that $TFq = FSq$, $TFVq = 1$; $UFq = FVq = (\cos + UVq \sin) TVq$; $F(Vq + \frac{\pi}{2} UVq) = UVq \cdot FVq$, $F(Vq + \pi UVq) = -FVq = (\cos - UVq \sin) (\pi - TVq)$; the function FVq is a *periodic* one, in the sense that it only changes *sign*, when we add $\pm \pi$ to TVq ; ANY VERSOR, Ur , may be considered as an *exponential function of a vector*, and put as such under the form FVq' , where the (positive) *tensor* TVq' shall not exceed π , and may therefore be treated

as the *angle of the versor*, $TVq' = \angle Ur$, with that *definite sense* of the word "angle," which was proposed in § XXXII.; if the versor Ur have been given, or found, under the form, FVq , and if $TVq > \pi$, whereas $TVq' \not> \pi$, it is proposed to consider Vq' , and *not* Vq , as the (principal) *value of the INVERSE EXPONENTIAL FUNCTION*, or to write $F^{-1}Ur = Vq'$; with this *definite signification* of that function we may therefore write, $\angle r = \angle Ur = TF^{-1}Ur$; also $UF^{-1}Ur = UVr = Ax \cdot r$, and $F^{-1}Ur = UVr \cdot \angle r$; we may also definitely interpret $F^{-1}Tr$ as $=ITr$ = that positive or negative number, or zero, which is the natural or Napierian *logarithm* of Tr ; and more generally we may agree to call the *inverse exponential function* (or the *IMPONENTIAL*) $F^{-1}r$, OF ANY QUATERNION r , the *LOGARITHM* of that quaternion, and to *interpret it definitely* as follows:

$$lr = F^{-1}r = F^{-1}Tr + F^{-1}Ur = ITr + UVr \cdot \angle r;$$

the *scalar of the logarithm* of a quaternion is thus the *logarithm* of the *tensor*, and the *vector of the logarithm* is the *logarithm of the versor*; in symbols,

$$Slr = ITr, Vlr = IUr = UVr \cdot \angle r$$

= *product of axis and angle*; that is, the *vector of the logarithm of any quaternion* is *constructed*, in our system, by the *REPRESENTATIVE ARC RECTIFIED*, and placed *PERPENDICULARLY TO THE PLANE*, or in the *DIRECTION OF THE AXIS*, of the quaternion; the *logarithm of a given quaternion*, thus interpreted, is *generally a DETERMINED quaternion*, but becomes *partially indeterminate*, when the given quaternion *degenerates* to a *negative number*, or to *zero*; we may agree to employ the usual symbol e^r , as a *concise expression* suggested by algebra (compare § XXIII.), for the *series* $1 + q + \frac{1}{2}q^2 + \&c.$, or for the *direct exponential function* Fq ; a *POWER* of a quaternion, with a *QUATERNION EXPONENT*, may then in general be *definitely interpreted* by means of the formula,

$$q^r = F(rF^{-1}q) = e^{r/q}; \text{ examples, } j^i = k, j^j = e^{-\frac{\pi}{2}};$$

expressions for the tensor and versor of the *general power*, q^r ; *MENTOR* of a quaternion, $Mq = ITq$ (this notation and nomenclature are not insisted on); *definite interpretation* of the *logarithm of a given quaternion to a given QUATERNION BASE*, namely, as the *quotient of their two natural logarithms*; $\log_q \cdot q' = lq' \div lq$; this *GENERAL LOGARITHM* might be so *interpreted* as to involve *two arbitrary integers*, as in some known theories; but we prefer, in this calculus, to *exclude such indetermination by definition*, in this as in other cases, wherever such exclusion is possible; interpretations of the *sine*, *cosine*, and *tangent*, of a quaternion; if we take *two arbitrary quaternions*, q and r , we shall still have, as in algebra,

$$e^r e^q = 1 + (r+q) + \frac{1}{2}(r^2 + 2rq + q^2) + \&c.;$$

but $r^2 + 2rq + q^2$, &c. will not in this calculus be equal to the *square*, &c., of $r+q$, unless $rq = qr$, or $V.VrVq = 0$, which will not generally happen; when this condition of *commutativeness*, of q and r as factors, is not satisfied, then if x be any scalar coefficient, supposed to vanish after the per-

formance of n successive differentiations, we shall indeed have *still* the expression,

$$\left(\frac{d}{dx}\right)^n . e^{xq} = r^n + nr^{n-1}q + \frac{1}{2}n(n-1)r^{n-2}q^2 + \dots + q^n;$$

but the polynome, thus obtained, will not be an expansion of the power $(r+q)^n$, Articles 537 to 550 ; Pages 545 to 557.

§ XCVI. A quaternion equation, $fq = r$, where f denotes a function of known form, may always be conceived as broken up into *four* equations of the *ordinary* algebraic kind, involving the four *constituents*, w, x, y, z , of the sought quaternion q (compare § LXXIV.) ; we may conceive xyz to be *eliminated* between these four equations, and the final equation in w to be resolved ; or we may suppose that $\rho = Vq$ is deduced (compare § xc.) from the vector equation, $Vfq = Vr$, and that its value is substituted in the scalar equation, $Sfq = Sr$, and that $w = Sq$ is then deduced therefrom ; or the elimination between these two equations, of vector and scalar kinds, may be performed in the opposite order ; we may also substitute, for the *one* vector equation, *three* scalar equations, such as

$$S . \kappa fq = S . \kappa r, S . \lambda fq = S . \lambda r, S . \mu fq = S . \mu r,$$

where κ, λ, μ are any arbitrary and auxiliary vectors ; equations of the form $\Sigma . bqa = c, \Sigma . a_2qa_1qa + \Sigma . b_1qb = c$, may be called respectively equations of the *first* and *second degrees* ; the *general equation of the n^{th} degree, in quaternions*, breaks up into four scalar equations which are *each of the same (n^{th}) degree* ; and *elimination* between these must be supposed to conduct, *generally*, to an ordinary equation of the degree of which the exponent is n^4 ; thus a *quadratic equation in quaternions* may be expected to have, *in general*, *sixteen roots*, or solutions, *at least of the symbolical kind* ; although in *particular cases*, by the vanishing of certain terms, the degree of the final equation may be depressed below its *general value*, . .

Articles 551 to 553 ; Pages 557 to 559.

§ XCVII. Discussion of the general equation of the first degree, $\Sigma . bqa = c$, where a, b, a', b', \dots and c are given quaternions, but q is a sought quaternion ; taking (compare § xcv.) the scalar and vector parts, and then eliminating w or Sq , there results a *linear and vector equation* of the form $\Sigma . \beta S . a\rho + V . r\rho = \sigma$, where $a, \beta, a', \beta', \dots$ and σ are given vectors, and r is a given quaternion, but ρ is a sought vector ; the equation gives

$$S . \lambda \sigma = S . \lambda' \rho, \text{ if } \lambda' = \Sigma . aS . \beta \lambda + V . s \lambda,$$

where $s = Kr$; forming similarly μ' from μ , and assuming λ and μ so that $V . \lambda \mu = \sigma$, we have

$m\rho = V . \lambda' \mu' = \Sigma V . a a' S . \beta' \beta \sigma + \Sigma V . a V (V . \beta \sigma . r) + SrV . \sigma r - VrS . \sigma r$, and the scalar coefficient $m = \Sigma S . a a' a'' S . \beta' \beta' \beta + \Sigma S (rV . a a' . V . \beta \beta) + Sr \Sigma S . r a \beta - \Sigma S . r a S . r \beta + SrTr^2$; remarks on the *notation* ; examples ; solutions of the equations, $V . \beta \rho a = \sigma, V . r\rho = \sigma$, agreeing with the results of § xc. ; discussion of the equation $bq + qb = c$, where b, c, q are quaternions ; one form of solution is, $2qSb = Vc + KbS . cb^{-1}$; another is, $2qb(b+b') = b'c + cb$, if $b' = Kb$, so that $b+b' = 2Sb$, and $bb' = b'b = Tb^2$;

or we may deduce and employ the equation, $(bq - qb)Sb = V.VbVc$; or may regard the proposed equation as a case of the following,

$$aq + qb = c,$$

which gives, $q(b^2 + 2bSa + Ta^2) = a'c + cb$, if $a' = Ka$; if we make $r = g + \gamma$, and $\Sigma \cdot \beta S \cdot a\rho + V \cdot \gamma\rho = \phi\rho$, $\psi = \phi + g$, the *general linear and vector equation* of the present section becomes $\psi\rho = \sigma$, and the problem of its solution comes to *inverting the function* ψ ; the *functional characteristic* ϕ is found to satisfy a SYMBOLIC AND CUBIC EQUATION, $0 = n + n'\phi + n''\phi^2 + \phi^3$, where n, n', n'' are three scalar coefficients, of which the values are assigned, in terms of the given vectors, $\alpha, \beta, \alpha', \beta', \dots$ and γ ; the characteristic ψ must therefore satisfy this *other* symbolic and cubic equation,

$$0 = \psi^3 - m''\psi^2 + m'\psi - m, \text{ where } m = g^3 - n''g^2 \\ + n'g - n, m' = 3g^2 - 2n''g + n', m'' = 3g - n'';$$

the *solution of the linear equation*, $\psi\rho = \sigma$, comes thus to be *found anew* under the form,

$$m\rho = m\psi^{-1}\sigma = (m' - m''\psi + \psi^2)\sigma = \sigma' - g\sigma' + g^2\sigma,$$

where σ' and σ'' are vectors derived from the given vector σ , by assigned operations, involving the given vectors $\alpha, \beta, \alpha', \beta', \dots$ and γ , but not the scalar g ; theorem of the PARALLELEPIPEDON OF DERIVATION, obtained by interpreting the lately written symbolic and cubic equation; for any proposed mode of LINEAR DEFORMATION, represented by the operation ψ , if we form the *three successive derivative lines*, $\psi\rho, \psi^2\rho, \psi^3\rho$, and then *decompose*, by projections, the original line ρ into three others, in these three directions, or in their opposites, the *ratio of each component to the corresponding derivative line will depend ONLY ON THE MODE OF DERIVATION*, and *not* generally on the *length*, nor on the *direction*, of the line ρ thus operated on; we have $m\psi^{-1}0 = 0$, and therefore generally $\psi^{-1}0 = 0$; but if it happen that g is a *root*, g_1 or g_2 or g_3 , of the *ordinary cubic equation*, $0 = m = g^3 - n''g^2 + n'g - n$, then the function $\psi\rho$ may vanish, without ρ itself vanishing; if, after assuming *any arbitrary vector* σ , we derive from it three others by the formulæ,

$$\rho_1 = \sigma'' - g_1\sigma' + g_1^2\sigma, \rho_2 = \sigma'' - g_2\sigma' + g_2^2\sigma, \rho_3 = \sigma'' - g_3\sigma' + g_3^2\sigma,$$

we shall have

$$\psi_1\rho_1 = \psi_2\rho_2 = \psi_3\rho_3 = m\sigma = 0;$$

that is, for these THREE DIRECTIONS, ρ_1, ρ_2, ρ_3 , we shall have

$$\phi\rho_1 = -g_1\rho_1, \phi\rho_2 = -g_2\rho_2, \phi\rho_3 = -g_3\rho_3;$$

this analysis might be developed so as to include the theories of the *axes of a surface of the second order*, and the *axes of inertia of a body*, . . .

Articles 554 to 567; Pages 559 to 569.

§ XCVIII. Definition of the DIFFERENTIAL of a FUNCTION of a quaternion,

$$dfq = \lim_{n \rightarrow \infty} .n \{f(q + n^{-1}dq) - fq\};$$

q and dq are here *any two quaternions*, Tdq being *not necessarily small*, but the positive whole number n being conceived to increase without limit; the *third* quaternion dfq , which results as the limit of this process, is a *function of the two assumed quaternions*, q and dq , of which the particular *form* depends on the form of the *proposed function*, f , but which is always *linear*, or *distributive*, with respect to the quaternion dq ; but this differential dfq is *not* in general reducible in *this* calculus, to a product of the form $f'q \cdot dq$, if $f'q$ denote a function of the quaternion q alone; when the function $f(q + dq)$ can be developed in a *series*, involving *terms* or parts of successively higher and *higher dimensions*, with respect to the quaternion dq , the *part* of this developement which is of the *first* dimension, relatively to dq , is (as in the ordinary differential calculus) the required differential dfq ; but it is proposed to *avoid*, in this calculus, adopting *this* as the *fundamental* property of a differential, because the recent *definition* can often be applied more easily than the *developement* can be found; examples; $d \cdot q^2 = q \cdot dq + dq \cdot q$, or more concisely, $d \cdot q^2 = qdq + dqd$, dq being treated as a *simple symbol*, or as if it were a *single letter*; $d \cdot q^{-1} = -q^{-1}dqq^{-1}$; in differentiating any *product* of quaternions, we simply differentiate each factor *in its own place*; we may *extend Taylor's series to quaternions*, under the form $f(q + dq) = e^f q$, where dq is treated as constant; examples; Articles 568 to 573; Pages 569 to 572.

§ CXIX. Geometrical applications; if a vector ρ be a given function ϕt of a variable scalar t , we may express its differential under the *usual* form, $d\rho = d\phi t = \phi' t \cdot dt = \rho' dt$, where $\rho' = \phi' t =$ a certain *derived vector*, which is *parallel to the tangent* to the *curve in space*, which is the *locus* of the extremity of ρ ; the *length* of this new vector is *unity*, $T\phi' t = 1$, if the *arc* be the independent variable; in mechanics, if t denote the *time*, and if a second differentiation have given $d\rho' = d\phi' t = \phi'' t \cdot dt = \rho'' dt$, then ρ' may be called the *vector of velocity*, and ρ'' the *vector of acceleration*, while ρ may be named the *vector of position*; in geometry, if t be again the arc of the curve, $\rho - \rho''^{-1}$ is the *vector of the centre of the osculating circle*, and ρ'' may therefore be called the *vector of curvature*; when a *surface* is expressed, as in § LXXXIX., by an equation of the form $f\rho = \text{const.}$, where f denotes a *scalar* function, we may then, by cyclical permutation under the sign S (see the same section LXXXIX.), express the *differentiated equation* of that surface under the form $df\rho = 2S \cdot \nu d\rho = 0$; the *logic* of this process will be more closely considered in § CI.; ν is a *NORMAL VECTOR*, and if we oblige it to satisfy the condition $S \cdot \nu\rho = 1$, then (compare § LXXXI.) its *reciprocal* ν^{-1} will represent, in length and in direction, the *perpendicular* let fall from the origin of vectors on the *tangent plane* to the surface, so that ν itself may be called, under the same conditions, the *vector of proximity*; without obliging ν to satisfy the equation $S \cdot \nu\rho = 1$, if we only choose it so as to give generally $S \cdot \nu d\rho = 0$, it will still be, as before, a *normal vector*, and this symbol ν may be used to form *EQUATIONS OF CLASSES OF SURFACES*; thus an *arbitrary cone* (with vertex at origin) may be denoted

by the equation $S \cdot \nu\rho = 0$, an *arbitrary cylinder* by $S \cdot \nu\alpha = 0$, and an *arbitrary surface of revolution* by $S \cdot \beta\nu\rho = 0$; this last equation is *analogous to an EQUATION IN PARTIAL DIFFERENTIALS*, and may be treated as such by a species of INTEGRATION, eliminating ν , and *introducing an arbitrary function*, under the form $\rho^2 = F(S \cdot \beta\rho)$, or $TV \cdot \rho\beta^{-1} = f(S \cdot \rho\beta^{-1})$, which last form was assigned in § LXIX.; conversely, by a process of *differentiation*, we can *eliminate the arbitrary function, f*, from this last equation, and so recover the formula of the present section, $S \cdot \beta\nu\rho = 0$, Articles 574 to 578; Pages 572 to 575.

§ c. *Geodetic lines*; the normal property of the osculating plane gives the following general equation of a geodetic, $S \cdot \nu d\rho d^2\rho = 0$, or $S \cdot \nu\rho'\rho'' = 0$, ρ being regarded as a function of some scalar variable; we have also this other *general* formula, $V \cdot \nu dU d\rho = 0$, where $dU d\rho$ denotes the differential of the versor of the differential of ρ , and is treated as a *simple symbol*; if we take the *arc* of the geodetic as the independent variable, or suppose that $Td\rho$ is constant, the last general form may be reduced to $V \cdot \nu d^2\rho = 0$, or $V \cdot \nu\rho'' = 0$; examples; geodetics on a *sphere*, and on an *arbitrary cylinder, cone*, and *surface of revolution*; VARIATIONS IN QUATERNIONS; formula for the *differential of the tensor* of an arbitrary vector σ , $dT\sigma = -S \cdot U\sigma d\sigma = S \cdot U\sigma^{-1} d\sigma$; this result will be extended in § cI.; $\delta d = d\delta$, $\delta f = f\delta$; the *variation of the length of the arc of a curve*, on any given surface, is expressed by the formula,

$$\delta f Td\rho = f\delta Td\rho = -\Delta S \cdot U d\rho \delta\rho + fS (dU d\rho \cdot \delta\rho);$$

hence the *varied equation of the surface* being $S \cdot \nu\delta\rho = 0$, the *general differential equation of a shortest line* is $V \cdot \nu dU d\rho = 0$, as above; *equations of limits*; for a geodetic on an *ellipsoid*, with the same significations of f and ν as in § LXXX., if $Td\rho$ be assumed as constant, the differential equation of the geodetic becomes,

$$0 = \frac{df d\rho}{2fU\rho} + S \frac{d\nu}{\nu}, \text{ and gives } T\nu \vee (fU d\rho) = \text{const.};$$

this reproduces the well-known theorem of Joachimstal, $P \cdot D = \text{const.}$, because $T\nu = P^{-1}$, and $\vee (fU d\rho) = D^{-1}$, if P be the *perpendicular* let fall from centre on tangent plane, and D the *semidiameter* parallel to the element $d\rho$; geodetic on a *developable surface*; proof of the *rectilinear form* which the *curve* assumes, when the *surface* is flattened into a *plane*; the general theorems of Gauss, respecting the *spheroidal excess* (or defect) of a *geodetic triangle* on an *arbitrary surface*, admit also of being proved by quaternions (see the investigation in § cVI.); reproduction of some geometrical properties, discovered by M. Delaunay, of the curve which on a *given surface*, and with a *given perimeter*, includes the *greatest area*; it is proposed to *name* a curve of this kind a DIDONIA; the *isoperimetrical formula* for its determination is

$$fS \cdot U \nu d\rho \delta\rho + c\delta f Td\rho = 0,$$

which gives the following *differential equation of a Didonia*,

$$c^{-1} d\rho = V.UvdUd\rho ;$$

geodetics are that *limiting case of Didonias*, for which the constant c is infinite ; in general, that constant may have its expression in various ways transformed, and may receive various geometrical interpretations ; among which the most remarkable is connected with the known property of the curve, that if a developable surface be circumscribed about a given surface, so as to touch it along a Didonia, and if this developable be then unfolded into a plane, the curve will at the same time be flattened generally into a circular arc, of which the radius = c . . . Articles 579 to 590 ; Pages 575 to 584

CI. More close examination of the *logic* (compare § XCIX.) of the process of *differentiating the equation* of a surface, and so obtaining the equation of its *tangent plane*, and the *normal vector* ν , without necessarily supposing for that purpose the differential $d\rho$ to be *small* ; differential of a *function of a function* of a quaternion ; $df(\phi q) = d(f\phi) q$; examples of the process ; case of the ellipsoid ; differentials of the *tensor* and *versor* of a quaternion, and of their *logarithms* : $dTq = S.dqUq^{-1}$, $dITq = S.dqq^{-1}$, $dIUq = dUqUq^{-1} = V.dqq^{-1}$; incidental notice of the general transformations, $r^{-1}(r^2q^2)^{\frac{1}{2}}q^{-1} = U(SrSq + VrVq) = U(rq + KrKq)$; by *inverting the function* which expresses (see § LXXIX.), the normal vector ν for the ellipsoid in terms of ρ , we find

$$\rho = (t^2 + \kappa^2) \nu - 2V.t\nu\kappa + 4(t - \kappa)^{-2} V.t\kappa S.t\kappa\nu ;$$

hence the equation of that other and *reciprocal ellipsoid*, on which ν terminates, may be thus written,

$$1 = S.\nu\rho = (t^2 + \kappa^2) \nu^2 - 2S.t\nu\kappa\nu + 4(t - \kappa)^{-2} (S.t\kappa\nu)^2 ;$$

the mean semi-axis of *this* reciprocal ellipsoid is b^{-1} (contrast § LXXXIV.) ; in general, the *locus of the extremity of the vector of proximity* (see § XCIX.), for *any surface*, may be very simply proved to be (as is otherwise known) a surface *reciprocal* thereto, by shewing that the equations

$$S.\nu\rho = c, S.\nu d\rho = 0, \text{ give } S.\rho\nu = c, S.\rho d\nu = 0, . . .$$

Articles 591 to 597 ; Pages 584 to 588.

§ CII. More close examination of the *extension* (§ XCVIII.) of *Taylor's Series* to quaternions ; proof that whenever the quaternion function $f(q + xr)$ can be developed, in a finite or infinite series, of the form $f_0 + xf_1 + x^2f_2 + \&c.$, x being a scalar, we must have $d^n f q = \Delta^n 0^n f_n$, if dq be treated as constant, and $= r$; other proof of this theorem, under the form that if $f(q + xdq) = f_0 + xf_1 + x^2f_2 + \&c.$, then $nf_n = df_{n-1}$; proof that if we suppose the n first of the successive differentials of the function of $f q$ to be *finite*, and if x be supposed *small* of the *first order*, then the expression $s_n = f(q + xdq) - f q - xdf q - \frac{1}{2}x^2d^2 f q - . . . - \frac{1}{2.3 . . n} x^n d^n f q$ is small of an *order higher than the n^{th}* ; or that not only the expression s_n itself, but

its n first successive *differential coefficients*, taken with respect to x , *vanish* with that scalar variable; it is to be remembered that q and dq are treated throughout *this* section (compare § XCIII.) as *two arbitrary quaternions*; and that Tdq is *not here* supposed to be *small*, although in *geometrical applications* it is often *convenient* to attribute small values to $Td\rho$; example from the equation of the *ellipsoid*, which may be *rigorously* developed under the *finite* form, $0 = f(\rho + d\rho) - f\rho = df\rho + \frac{1}{2}d^2f\rho$, $d\rho$ denoting an *arbitrary chordal vector*, drawn from the extremity of ρ , to *any other point* of the surface, Articles 598 to 601; Pages 588 to 592.

§ CIV. When $d\rho$ is thus treated as a finite and chordal vector, the equation

$$0 = df\rho + \frac{1}{2}d^2f\rho, \text{ or } 0 = 2S \cdot \nu d\rho + S \cdot d\nu d\rho,$$

or the same equation with an additional term $S \cdot \nu d\rho S \cdot \omega d\rho$, where ω is an arbitrary vector, represents an ellipsoid, or other surface of the second order, which *osculates* in *all* directions to the given surface $f\rho = \text{const.}$, or has with it *complete contact of the second order*, at the extremity of ρ ; if σ be the vector of the centre of the *sphere* which osculates to the surface in the *direction* marked by the limiting value of $Ud\rho$, then $\frac{\nu}{\rho - \sigma} = S \frac{d\nu}{d\rho}$, the second member being regarded as a function of this value of $Ud\rho$; applied to the ellipsoid, this formula reproduces the known expression $D^2 \cdot P^{-1}$, as the value for $T(\rho - \sigma)$, or for the radius of curvature of a normal section of the surface,

Articles 602 to 606; Pages 592 to 596.

§ CIV. For any surface, $S \cdot \delta d\nu d\rho = S \cdot d\nu \delta d\rho$, if in forming $\delta d\nu$ we operate only on $d\rho$, but not on ρ itself, as contained in the expression of $d\nu$; hence it may be inferred that the directions of osculation of the *greatest and least spheres*, determined by the formula $\delta S \cdot \nu d\rho^{-1} = 0$, are also the directions of the *lines of curvature*, for which consecutive normals intersect, and which have for their differential equation $0 = S \cdot \nu d\nu d\rho$; this latter equation expresses that $d\rho \perp V \cdot \nu d\nu$, and therefore contains one of the theorems of Dupin, namely, that the tangent to a line of curvature on any surface at any point is *perpendicular* to its *conjugate tangent*; equations of the *indicatrix*, $S \cdot \nu d\rho = 0$, $S \cdot d\nu d\rho = \text{constant}$; the equation of the lines of curvature may also be thus written, $0 = S \cdot d\nu \delta U d\rho$; or thus, $0 = V \cdot d\rho dU \nu$; this last form contains a theorem of Mr. Dickson, that if two surfaces cut along a *common line of curvature*, they do so *at a constant angle*; transformation of the equation of § CIV., for the curvature of a section of a surface,

$$\frac{\nu}{\sigma - \rho} = S \frac{\nu d^2\rho}{d\rho^2} = S \frac{\nu}{\omega - \rho},$$

conducting to the theorem of Meusnier; other general transformation and interpretation of the formula of § CIV., for the curvature of a normal section; if on the normal plane CPF' to a given surface, containing a given linear element PP' , we project the normal to the surface at the *near point*,

r' , this projected normal will cross the given normal CP , which is drawn at the given point P , in the centre C of the sphere which osculates to the surface along the element, Articles 607 to 612; Pages 596 to 601.

§ cv. Considering the vector ρ , of a variable point on any surface, as a function, = $\psi(x, y)$, of two scalar variables, x and y , which are themselves regarded as functions of some one independent and scalar variable, we may write,

$$\begin{aligned} d\rho &= \rho' dx + \rho'' dy; \quad d\rho' = \rho''' dx + \rho'''' dy; \quad d\rho'' = \rho'''' dx + \rho'''''' dy; \\ d^2\rho &= \rho'' dx^2 + 2\rho''' dx dy + \rho'''' dy^2 + \rho'''' dx + \rho'''''' dy; \\ \rho', \rho'', \rho''', \rho''', \rho'''' &\text{ being five new vectors;} \end{aligned}$$

it is allowed to write $\nu = V. \rho' \rho''$, because ρ' and ρ'' are tangential, and therefore the ν thus found is normal; in the expression for $S. \nu d^2\rho$, d^2x and d^2y disappear; and if we make $U\nu (\sigma - \rho)^{-1} = R^{-1}$, so that R is the radius of curvature of a normal section, of which σ is the vector of the centre of curvature, we shall have, by § civ., an equation of the form,

$$0 = R^{-1} d\rho^2 - S. U\nu d^2\rho = Adx^2 + 2Bdx dy + Cdy^2;$$

for a line of curvature, we have

$$0 = Adx + Bdy = Bdx + Cdy, \text{ and therefore } AB - C^2 = 0,$$

where

$$A = R^{-1} \rho'^2 - S. \rho'' U\nu, \quad B = R^{-1} S. \rho' \rho'', \quad C = R^{-1} \rho''^2 - S. \rho'' U\nu;$$

R_1, R_2 being the two extreme radii of curvature, the MEASURE OF CURVATURE of the surface may be thus expressed,

$$R_1^{-1} R_2^{-1} = S. \frac{\rho''}{\nu} S. \frac{\rho''}{\nu} - \left(S. \frac{\rho''}{\nu} \right)^2;$$

example; deduction of the usual formula, $(rt - s^2) (1 + p^2 + q^2)^{-2}$; in general if $e = -\rho'^2, f = -S. \rho' \rho'', g = -\rho''^2$, so that the square of the length of a linear element of the surface has for expression

$$Td\rho^2 = edx^2 + 2fdx dy + gdy^2,$$

the recent expression for the measure of curvature is shewn to depend only on the three scalars e, f, g , on their six partial differential coefficients of the first order, and on three of their nine partial differential coefficients of the second order, taken with respect to x and y ; in this way is reproduced by quaternions a very remarkable theorem of Gauss, namely, that if a surface be treated as an infinitely thin and flexible, but inextensible solid, and be then TRANSFORMED as such into another surface, such that each LINEAR ELEMENT of the new is equal in length to the corresponding element of the old one, the MEASURE OF CURVATURE at each point will NOT BE ALTERED by this TRANSFORMATION,

Articles 613 to 615; Pages 601 to 604.

§ cvi. If x denote the length of the geodetic line AP , drawn on the surface from a

fixed point A , and if y denote the angle BAP which the variable geodetic AP makes there with a fixed line AB , then

$$\rho^2 = -1, \quad S. \rho' \rho = 0, \quad \text{or } e = 1, \quad f = 0,$$

and these equations may be differentiated; hence if we make $m = \sqrt{g} = T\rho$, the general expression for the measure of curvature reduces itself to the following, which (with a somewhat different notation) was first discovered by Gauss,

$$R_1^{-1}R_2^{-1} = -m''m^{-1}; \quad \text{or, } R_1^{-1}R_2^{-1} = d^2T\delta\rho \div (d\rho^2T\delta\rho);$$

treating x and y as functions of the arc s of a new geodetic on the surface, *not* drawn from the fixed point A , and denoting by v the angle between an element ds or PP' of this new geodetic, and the prolongation of the old geodetic line AP , the differential equation of the new geodetic becomes,

$$x'' = mm'y'^2, \quad \text{or } v' = -m'y', \quad \text{or } dv = -m'dy;$$

we may also conveniently write, in a slightly modified notation,

$$\delta v = -m'\delta y, \quad \text{or } \delta v = -dT\delta\rho \div Td\rho,$$

d referring here to motion *along* the original geodetic AP , and δ to passage *from* that line to a near one; $d\delta v$, or $-m''dx\delta y$, is then a symbol for the *spheroidal excess* (compare § c.) of a little geodetic quadrilateral, of which the area = $mdx\delta y$; *dividing the excess by the area*, we find the quotient = $-m''m^{-1}$ = the measure of curvature of the surface; but also this *measure* = $R_1^{-1}R_2^{-1} = V. dU\nu\delta U\nu \div V. d\rho\delta\rho$ = the area of the *corresponding* superficial element of the unit-sphere, divided by the element of area of the given surface, this correspondence consisting in a *parallelism* between radii and normals; hence, as Gauss proved, the **TOTAL CURVATURE** of any small or large *closed figure*, on any arbitrary surface, bounded by geodetic lines, or the area of the corresponding portion of the surface of the unit-sphere (not generally bounded by great circles), is equal (with a proper choice of units) to the **SPHEROIDICAL EXCESS of the figure**; *singular points* are here excluded, and the *sign* of the element of the spherical area is supposed to *change*, when we pass from a *convexo-convex* to a *concavo-convex* surface, Articles 616 to 619; Pages 604 to 609.

§ CVII. Many other geometrical applications of differentials of quaternions might easily be given; for instance, they serve to express with ease what M. Liouville has called the *geodetic curvature* of a curve upon any surface; they may also be employed to calculate the *normal* and *osculating planes*, and the *evolutes*, *torsions*, &c. of curves of double curvature; transformations of the symbols $\triangleleft \triangleleft'$, \triangleleft^2 , where

$$\triangleleft = \frac{id}{dx} + \frac{jd}{dy} + \frac{kd}{dz}, \quad \triangleleft' = \frac{id}{dx'} + \frac{jd}{dy'} + \frac{kd}{dz'},$$

$x y z x' y' z'$ being six independent and scalar variables; the formulæ,

$$\triangleleft (it + ju + kv) = - \left(\frac{dt}{dx} + \frac{du}{dy} + \frac{dv}{dz} \right)$$

$$+ i \left(\frac{dv}{dy} - \frac{du}{dz} \right) + j \left(\frac{dt}{dz} - \frac{dv}{dx} \right) + k \left(\frac{du}{dx} - \frac{dt}{dy} \right),$$

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = -\Delta^2 v,$$

appear likely to become hereafter important in mathematical physics; $-\Delta v$ may represent the *flux of heat*, if v be the *temperature* of a body; or if v be the *potential* of a system of attracting bodies, then Δv represents, in amount and in direction, the *accelerating force* which they exert at the point xyz ; in geometry, the vector Δv is *normal* to the *surface* for which the scalar function $v = \text{constant}$; when operating on such a function,

$$\Delta = - (S. d\rho)^{-1} d, \dots$$

Article 620; Pages 609 to 611.

§ CVIII. Applications of quaternions to *physical astronomy*; the vector function, $\phi\alpha = \alpha^{-1} T\alpha^{-1}$, may be called the **TRACTOR** of α , because it represents, in length and in direction, the accelerating force of attraction which an unit of mass at the origin exerts on a point placed at the end of the *vector of position*, α ; by the rules of this calculus, this function may be thus transformed,

$$\phi\alpha = dU\alpha \div V. a\alpha = (U\alpha)' \div V. \alpha\alpha';$$

the differential equation of *motion of a binary system*, $d^2\alpha = M\phi\alpha dt^2$, or $\alpha'' = M\phi\alpha$, gives the following integrals of the first order, $V. \alpha\alpha' = \gamma$, $\alpha' = M\gamma^{-1}(\epsilon - U\alpha)$, where γ and ϵ are constant vectors, but α is a variable vector; the first contains the laws of *constant plane and area*, and the second contains the **LAW OF THE CIRCULAR HODOGRAPH**; eliminating the vector of velocity, α' , we obtain this *equation of the orbit*, $0 = T\alpha + S. \alpha\epsilon + M^{-1}\gamma^2$, or $r^{-1} = p^{-1}(1 + e \cos v)$, agreeing with a well-known result respecting the *conic-section form* of the curve, and *focal character* of that body about which the other is conceived to move; the varying tangential *velocity* of this latter body may be *decomposed into two parts, both constant in amount, and one constant also in direction*; theorem of **HODOGRAPHIC ISOCHRONISM**, corresponding to Lambert's theorem; allusion to a conception of Moebius; the difference $\phi(\alpha + \Delta\alpha) - \phi\alpha$, or $\Delta\phi\alpha$, of the tractor function $\phi\alpha$, might perhaps be called the **TURBATOR**, because it expresses, with Newton's law, the amount and direction of the *disturbing force* which an unit-mass, supposed to be situated at the common origin **B** of the two vectors α and $\alpha + \Delta\alpha$, exerts on a body **A** situated at the end of the latter variable vector, to disturb its relative motion about a body **C** at the end of the former vector; developement of this disturbing force, under the supposition that $T\Delta\alpha < T\alpha$, or that the distance $b = \overline{CA}$, of the disturbed body **A** from the centre **C** of the relative motion, is less than the distance $a = \overline{BC}$ of the disturbing body **B** from the same centre; example, where **A**, **B**, **C** denote moon, sun, and earth; we have the transformation,

$$\phi(\beta + \alpha) = (1 + q)^{-\frac{1}{2}} (1 + q')^{-\frac{3}{2}} \phi\alpha, \text{ if } q = \beta\alpha^{-1}, q' = Kq = \alpha^{-1}\beta;$$

hence results a development of the form $\phi(\beta + \alpha) = \sum_{n, n'} \phi_{n, n'}$, in which the law of formation of the terms is assigned; the sun's disturbing force on the moon is in this way seen to admit of being decomposed into a *series of groups of smaller and smaller forces*, in the varying plane of the three bodies, represented in amount and in direction by the terms of this development; if a , b denote the geocentric distances of the sun and moon, and C their geocentric elongation measured from the sun towards the moon in their common great circle in the heavens, then the *angle* from the sun's geocentric vector $-a$ to the component force $\phi_{n, n'}$ is $= (n - n') C$, and the *intensity* of the same partial force is $= m_{n, n'} (ba^{-1})^{n+n'} a^{-2}$, $m_{n, n'}$ being an assigned and rational numerical coefficient; in the first and principal group, there are *two* component forces, of which one, $\phi_{1, 0}$, has its intensity $= \frac{1}{2}ba^{-3}$, if the sun's mass be taken for unity, and is directed along the moon's geocentric vector β prolonged, or towards the moon's apparent place in the heavens, while the other, $\phi_{0, 1}$, has an *exactly triple intensity*, and is directed towards what may be called a *fictitious moon*, or to a point which is a sort of *reflexion* of the moon's place with respect to the sun; the second group contains *three* partial forces, which may be said to be directed towards *three suns* (one real and two fictitious), and the intensities of these forces, taken in a suitable order, are *exactly proportional to the whole numbers 1, 2, 5*; these results may be indefinitely extended, and applied to the perturbation of an inferior by a *superior planet*, &c.; some of these and other results of the application of quaternions to *mechanical* or *physical* problems, such as the *conditions of equilibrium*, the theory of *statical couples*, and the *motion of a system of mutually attracting bodies*, were communicated to the Royal Irish Academy in 1845; the present writer has since made *other physical applications* of the same principles, and has published some of them, but is aware that nothing important in that way is likely to be done, until the more full co-operation of other and better mathematicians shall have been gained,

Articles 621 to 624; Pages 611 to 620.

§ CIX. A DEFINITE INTEGRAL in quaternions may be interpreted as a *limit of a sum*; but, *even* when the function to be integrated remains *finite* between the limits of integration, *still* if the differential factor dq under the sign of integration be *itself* essentially a *quaternion*, then a certain degree of *indetermination* of value of the quaternion integral $\int_{q_0}^{q_1} F(q, dq)$ arises from the possibility of assuming indefinitely many different *laws of dependence* of the variable quaternion q on a *scalar* variable t , which latter may be supposed to change from 0 to 1, while q changes from one given quaternion value q_0 to another q_1 ; in this way arises a *new sort of variation of a definite integral*, depending on the *non-commutative* character of multiplication, which may be symbolized by the formula,

$$\delta Q = \delta \int_{q_0}^{q_1} F(q, dq) = \int_{q_0}^{q_1} \{ \delta_q F(q, dq) - d_q F(q, \delta q) \};$$

for example,

$$\delta \int f q dq = \int (\delta f q \cdot dq - df q \cdot \delta q), \text{ if } \delta q_0 = 0, \delta q_1 = 0;$$

more particularly,

$$\delta \int_{q_0}^{q_1} q dq = \delta \int_0^1 q_t q'_t dt = \int_0^1 (\delta q_t q'_t - q'_t \delta q_t) dt,$$

the integral relatively to t being interpreted as the limit of a sum; examples of *different functional forms* which may be assumed for q_t , and of the *different quaternion values* thereby obtained for the integral $\int_{q_0}^{q_1} q dq$; this sort of *variation* of a definite integral *vanishes*, as in the ordinary integral calculus, when the function $F(q, dq)$ is an *exact differential*; for example, although, between given quaternion limits, the integrals of $q dq$ and dqq are each *separately* subject to the kind of indetermination above explained, yet the integral of their *sum* is fixed, and we may write, *definitely*, as in algebra,

$$\int_{q_0}^{q_1} (q dq + dqq) = q_1^2 - q_0^2;$$

analogous remarks would apply to such expressions as

$$R = \int_{r_0}^{r_1} \int_{q_0}^{q_1} F(q, r, dq, dr);$$

if the subject of this section shall be hereafter pursued, it will be proper to combine it with the researches of M. Cauchy, respecting definite integrals taken between *imaginary limits* of the ordinary kind, and respecting that *other* species of *indetermination*, which arises from the passage of functions through *infinity*, and *not* from any supposed absence of the *commutative* property of multiplication, . . . Articles 625 to 630; Pages 620 to 627.

§ cx. Differentiation of *implicit functions*, and of *radicals*; examples; if fx denote any *scalar function* of a *scalar variable* x , and if $dfx = f'x dx$, then in passing to *quaternions*, we have $V.VqVfq = 0$; if also we suppose $UVfq = +UVq$, and denote by $dq - \delta q$ that *part* of dq which is a vector perpendicular to Vq , we shall have, under these conditions, the formula $dfq = f'q \delta q + TVfq \cdot dUVq$, which may be in various ways transformed, and gives the equation,

$$Vq \delta f q + df q V q = f' q (V q dq + dq V q);$$

connexion of *differentials* and *developements* with *equations of the first degree*; to find the *differential of the square root of a quaternion* r , we are by § xcvi. to resolve the equation $qdq + dq q = dr$, which is of the same form as the equation $bq + qb = c$, discussed in § xcvi.; and a *series of equations* of this *linear form* may be employed to *develope the square root of a sum*, in a *quaternion series*, of the form

$$(b^2 + c)^{\frac{1}{2}} = b + q_1 + q_2 + \&c., \quad$$

Articles 631 to 635; Pages 627 to 631.

§ cx. *Quadratic equations in quaternions* (compare § xcvi.); an equation of the form $q^2 = qa + b$, or of this connected form, $r^2 = ar + b$, where $abqr$ are

quaternions, and $q + r = a$, $qr = -b$, has in general SIX ROOTS, of which two are real, and four imaginary; the determination of these six quaternion roots depends on a scalar equation of the sixth degree, which is of cubic form; the scalar and cubic equation thus obtained has in general one positive and two negative roots; case in which one root of the cubic vanishes; examples of the above form of a quadratic equation in quaternions,

$$q^2 = 5qi + 10j, \quad q^2 = qi + j;$$

more general example, $q^2 = qa + \beta$, where α and β denote two rectangular vectors, $S\alpha = 0$, $S\beta = 0$, $S.\alpha\beta = 0$; the six quaternion roots of this last quadratic are given by the three formulæ,

- I. $q = \frac{1}{2}\alpha + \alpha^{-1}\beta \pm \frac{1}{2}\alpha^{-1}(\alpha^4 + 4\beta^2)^{\frac{1}{2}}$,
- II. $q = \frac{1}{2}(1 + U\beta)\{\alpha \pm (\alpha^2 + 2T\beta)^{\frac{1}{2}}\}$,
- III. $q = \frac{1}{2}(1 - U\beta)\{\alpha \pm (\alpha^2 - 2T\beta)^{\frac{1}{2}}\}$,

in which it is to be remembered that $\alpha\beta = -\beta\alpha$, so that the ordinary rules of algebra are not all applicable here (§§ X., XI., &c.); by the peculiar rules of the present calculus, it is easy to shew that the common value of q^2 and $qa + \beta$ is, for the first formula,

$$\frac{1}{2}\alpha^2 \pm \frac{1}{2}(\alpha^4 + 4\beta^2)^{\frac{1}{2}};$$

each of the other two formulæ may also be shewn, *à posteriori*, to give equal values for the two quaternions q^2 and $qa + \beta$; the third formula gives always two imaginary values for q ; but, according as $\alpha^4 + 4\beta^2 <$ or > 0 , we shall have two real quaternions from the second formula, and two imaginary vectors from the first, or two real vectors from the first, and two imaginary quaternions from the second expression; in the former case, the two real quaternion roots of the quadratic equation have a common tensor = $\sqrt{T\beta}$; in the latter case, the two real vector roots have unequal lengths, or tensors, but $\sqrt{T\beta}$ is still the geometrical mean between them; the distinction between these two cases corresponds (compare § LXXVII.) to the imaginariness or reality of the intersections of the sphere, $\rho^2 = S.ap$, and the right line, $V.ap = \beta$; the IMAGINARY QUATERNIONS considered in the present section (compare § XCVI.) are all reducible to the form, $q = q' + q''\sqrt{-1}$, where q' and q'' are quaternions of the real and ordinary kind, such as have been hitherto considered in these Lectures, and $\sqrt{-1}$ is the old and ORDINARY IMAGINARY SYMBOL of common algebra, and is to be treated, in this sort of combination with the peculiar symbols, (*ijk*, &c.) of the present calculus, not as a real vector (contrast the earlier use of the same symbol in § XXXV.), but as an imaginary scalar; an expression of this mixed form, $q' + \sqrt{-1}q''$, is named by the writer a BIQUATERNION; the study of them will be found to be important, and indeed essential, in the future development of this calculus,

Articles 636 to 650; Pages 631 to 643.

§ CXII. Application of the foregoing principles, to continued fractions, of the form

$$u_x = \left(\frac{b}{a} \right)^x c,$$

where a , b , and c ($= u_0$) are any three given quaternions, and x is a positive whole number; making

$$v_x = (u_x + q_2)(u_x + q_1)^{-1},$$

we have $v_x = q_2^x v_0 q_1^{-x}$, where q_1 , q_2 are any two roots of the quadratic equation $q^2 = qa + b$; examples,

$$\left(\frac{j}{i+} \right)^x 0, \left(\frac{j}{i+} \right)^x c, \left(\frac{10j}{5i+} \right)^x c, \left(\frac{\beta}{\alpha+} \right)^x \rho_0;$$

in the two first of these four examples, the continued fraction has generally a *period of six values*, which may be found at pleasure by employing the two *real quaternion roots* of the quadratic equation $q^2 = qi + j$, namely,

$$q_1 = \frac{1}{2}(1 + i + j - k), q_2 = \frac{1}{2}(-1 + i - j - k);$$

or two *conjugate imaginary solutions* of that quadratic, such as the pair

$$q_1 = zi - k, q_2 = z^{-1}i - k, \text{ where } z = (\cos + \sqrt{-1} \sin) \frac{\pi}{3}, \sqrt{-1} \text{ being the old}$$

imaginary symbol (compare § CXI.); or the *other pair* of imaginary roots of the same quadratic equation, included in the expression,

$$q = \frac{1}{2}(i + k) \pm \frac{1}{2}(1 - j)\sqrt{-3};$$

or by any other selection of two roots, for instance, by combining one real and one imaginary root; the six real quaternion terms of the period, found by any of these combinations of roots, agree with those obtained by actually performing the *divisions* prescribed by the form of the continued fraction; in the third example above cited, of such a fraction, the value does not circulate, but (generally) converges to a limit, so that

$$\left(\frac{10j}{5i+} \right)^\infty c = 2k - i, \text{ unless } c = 2k - 4i;$$

in this last case, and also in the case when $c = 2k - i$, that is, when c is a real root of the quadratic $c^2 + 5ci = 10j$, the value of the fraction is constant; geometrical interpretations, for the case where $c = ix_0 + kz_0$, x_0 and z_0 being regarded as the coordinates of an assumed point P_0 in the plane of ik (or xz); successive derivation of other points P_1 , P_2 , &c., according to a law assigned; if the assumed point be placed at either of two fixed points F_1 , F_2 , in the same plane of ik , its position will not be changed by this mode of successive derivation; but if P_0 be taken anywhere else in the plane, the derivative points will indefinitely tend to the fixed position F_2 , so that we may write

$$P_\infty F_2 = 0, P_\infty = F_2, \text{ unless } P_0 = F_1;$$

law of this approach; continual bisection of the quotient, $PF_2 \div PF_1$, of the distances of the variable point P from the two fixed points; theorem of the two circular segments, on the common base F_1F_2 , and containing the

two sets of alternate and derivative points, $P_0, P_2, P_4 \dots$ and $P_1, P_3, P_5 \dots$ to infinity; verification by co-ordinates; relation between the two segments; more general geometrical theorems of the same kind, obtained as interpretations of the results of calculation with quaternions, respecting the fourth example of a continued fraction above mentioned, with the supposition that β is a vector perpendicular to α and to ρ_0 , and under the condition

$$\alpha^4 + 4\beta^2 > 0 \text{ (see again § CXI.)};$$

interpretation of this condition; when $\alpha^4 + 4\beta^2 < 0$, there is no tendency of the variable point to converge to any fixed position; the quadratic $q^2 = qa + \beta$ (of § CXI.) gives

$$q^4 = q^2\alpha^2 + \beta^2, (2q^2 - \alpha^2)^2 = \alpha^4 + 4\beta^2;$$

but when $\alpha^4 + 4\beta^2 = 0$, the biquaternion solutions of the quadratic give, indeed, like the real roots,

$$(2q^2 - \alpha^2)^2 = 0, \text{ but not, like them, } 2q^2 - \alpha^2 = 0;$$

those solutions give in this case $2q^2 - \alpha^2 = 4SqVq$, $Vq = \rho' \pm \sqrt{-1} \rho''$, where ρ' and ρ'' denote two real and rectangular and equally long vectors; and the square of such an expression vanishes, without our being allowed to equate the expression itself to zero; algebraical interpretation of the general results at the commencement of this section, divested of quaternion symbols, and connected with a functional law of derivation of four scalars from four other scalars arbitrarily assumed, and from eight given and constant scalars; the indefinite repetition of this process of derivation conducts generally to one ultimate or limiting system, of four derivative scalars, Articles 651 to 668; Pages 643 to 664.

§ CXIII. A biquaternion may be considered generally as the sum of a biscalar and a bivector; we may also conveniently introduce biconjugates, bitensors, and bivectors, and establish general formulæ for such functions or combinations of biquaternions, which shall be symbolical extensions of earlier results of this calculus; thus, in any multiplication, the bitensor of a product can only differ by its sign from the product of the bitensors; there exists an important class of biquaternions, for which the bitensors vanish; such biquaternions may be called nullifiers, or nullifiers, because each may be associated (indeed in infinitely many ways), as multiplier or as multiplicand, with another factor different from zero, so as to make their product vanish (compare § CXII.); general expressions for the reciprocal of a biquaternion; the reciprocal of a nullifier is infinite; a real quaternion has generally a pair of imaginary, as well as a pair of real square roots; hints respecting the geometrical utility of the biquaternions, in transitions (for example) from closed to unclosed surfaces of the second degree, and in other imaginary deformations; reference to a proposed Appendix to these Lectures, containing a geometrical translation of an investigation so conducted, respecting the inscription of gauche polygons, in ellipsoids, and in hyperboloids, Articles 669 to 675; Pages 664 to 674.

§ cxiv. Brief outline of the quaternion *analysis* employed in such researches respecting the inscriptions of polygons in surfaces (with which are connected other problems respecting the circumscriptions of polyhedra); *equation of closure*, resumed from § LV.; *distinction* between the cases of even-sided and odd-sided polygons; if it be required to inscribe in a given sphere, or other surface of the second order, a gauche polygon with an *odd* number of sides, passing successively through the *same* number of given points, there exists in general *one real chord of solution*, determining *two real* OR *imaginary positions* of the *initial point* of the polygon; but, if the polygon be *even*-sided, there are then (for the sphere, ellipsoid, or double-sheeted hyperboloid) *two real chords of real AND imaginary solution*; for the single-sheeted hyperboloid (see Appendix), these two chords *may* themselves become *imaginary*; in general they are *reciprocal polars* of each other; thus there may in general be inscribed, in a surface of the second order, two real OR two imaginary gauche polygons, with an *odd* number of sides, passing through as many given and non-superficial points; whereas, if the surface be *non-ruled*, and if the number of points and sides be *even*, there may in general be inscribed *two real, and two imaginary polygons*, which become *all four real, or else all four imaginary*, when we pass to a *ruled* surface; if we conceive that the inscribed gauche polygon $PP_1 \dots P_n$ has $n + 1$ sides, of which *only the first n* are obliged to pass through so many given and non-superficial points, $A_1, \dots A_n$, then the *closing side*, or *final chord*, P_nP , belongs to a certain *system of right lines in space*, of which it is interesting to study the *arrangement*; quaternion formulæ connected therewith; when the number n of the given points is *even*, so that the number $n + 1$ of the sides of the polygon is odd, the *closing chords touch two distinct surfaces of the second order*, which have *quadruple contact with the original surface, and with each other*, and are geometrically related to each other and to the given surface, as are *three single-sheeted hyperboloids* which have *two common pairs of generatrices*; when the number of the given points is *odd*, or of the sides of the polygon even, then the *envelope of the closing side* consists of a *pair of cones*, which are *imaginary* if the given surface be non-ruled, but may become *real* by *imaginary deformation*, namely, by passing to the case of inscription in a *ruled* surface; in this last case, the lines on the surface, which are *analogous to lines of curvature*, as being those linear loci of the initial point P , which are *bases of developable surfaces* composed by corresponding systems of positions of the variable chord PP_n , are *rectilinear generatrices* of the given surface; these *bases* become *imaginary*, when we return to the *sphere*, ellipsoid, or other non-ruled surface, as that in which the polygon is to be inscribed; when the number of given points is even, the *tangents* to the *two corresponding curves* on the given surface, at any proposed point P , are *conjugate*, being *parallel to two conjugate diameters*; there exist also certain *harmonic relations* between the lines and planes which enter into this theory of *inscription*; references to communications by the present writer, on this subject, of which some have been already published, (see also Appendix B), Articles 676, 677; Pages 674 to 678.

§ cxv. More full discussion of the signification of an equation, namely,

$$V. \rho\alpha = \rho V. \rho\beta, \text{ or } V. \alpha\rho = \rho V. \beta\rho,$$

which had presented itself in the foregoing analysis ; this equation represents generally a certain *curve of double curvature* which is of the *third order*, as being *cut by an arbitrary plane in three points*, real or imaginary ; this curve is the *common intersection* of a certain *system of surfaces of the second order* ; it intersects the *sphere* $\rho^2 = -1$ in *two real and two imaginary points*, namely, in the initial positions of the first corner of an inscribed and even-sided polygon (§ cxiv.), but it may be said also to intersect the same sphere in *two other imaginary points, at infinity* ; if we confine ourselves to *real* vectors and quaternions, we can express a variety of *other geometrical loci* by equations of remarkable simplicity ; interpretations of the ten equations,

$$\begin{aligned} q = 0, q = 1, q = -1, Tq = 1, Uq = 1, Uq = -1, \\ Vq = 0, Sq = 0, Sq = 1, Sq = -1, \text{ where } q = (\rho\alpha^{-1})^2; \end{aligned}$$

with the same meaning of q , if $\beta \perp \alpha$, the equation $Vq = \beta$ represents a certain *hyperbola* ; if $\alpha\beta\gamma$ denote three real and rectangular vectors, the equation $(\gamma V. \alpha\rho)^2 + (\gamma V. \beta\rho)^2 = 1$ represents a certain *ellipse* ; the equation $(S. \alpha\rho)^2 + (\gamma V. \alpha\rho)^2 = 1$ denotes the *system of an ellipse and an hyperbola*, with one *common pair of summits*, but situated in *two rectangular planes* ; an equally simple equation can be assigned representing a *system of two ellipses*, in two rectangular planes, having a common pair of summits ; the equation $\iota\rho\kappa\rho = \rho\kappa\rho\iota$, or $V. \iota\rho\kappa\rho = 0$, represents a system of *two rectangular right lines*, bisecting the angles between ι, κ ; while the equation $\iota\rho\kappa\rho = \rho\iota\rho\kappa$, or $0 = V. \rho V. \iota\rho\kappa$, represents a system of *three rectangular lines*, namely, these two bisectors, and a line perpendicular to their plane ; example from the ellipsoid, equation $V. \nu\rho = 0$; general equation of surfaces of the second order ; equation of Fresnel's *wave-surface* ; general formulæ for translating any equation in co-ordinates into an equation in quaternions,

$$x = -iS. i\rho, y = -jS. j\rho, z = -kS. k\rho ;$$

other expressions for geometrical loci may be obtained, by regarding ρ as the *vector part* of a *variable quaternion* q , which is obliged to satisfy some given equation, while its *scalar part* w is variable ; formulæ may be assigned which shall represent, respectively, on this plan, what may be called a *full circle*, and *full sphere*, Articles 678, 679 ; Pages 678 to 688.

§ cxvi. Equation of the *focal hyperbola*, $V. \eta\rho \cdot V. \rho\theta = (V. \eta\theta)^2$, resumed from § LXXXVIII. ; proof of the *adequacy* of this *one* equation to represent that *curve* ; geometrical illustrations of the significations of the two constant vectors η and θ ; they are the two oblique *co-ordinates of an umbilic* of the ellipsoid, referred to the asymptotes of the focal hyperbola, when *directions* as well as lengths are attended to ; other elementary geometrical illustrations and confirmations of some of the results of earlier sections (especially of §§ LXXXVI. to LXXXVIII.), chiefly as regards the equations in-

volving η, θ ; additional calculations and interpretations, designed principally as *exercises in quaternions*; introduction of the two new vectors,

$$\lambda_1 = \rho - 2(\eta + \theta)^{-1} \mathbf{S} \cdot \theta \rho, \quad \epsilon = 2\mathbf{V} \cdot \eta \theta \mathbf{T}(\eta + \theta)^{-1},$$

with three other vectors $\lambda_2, \lambda_3, \lambda_4$, determined in terms of ρ by expressions analogous to that for λ_1 ; we have the equations,

$$\begin{aligned} \mathbf{T}(\lambda_1 - \epsilon) &= \mathbf{b} + \mathbf{b}^{-1} \mathbf{S} \cdot \epsilon \rho, \quad \mathbf{T}(\lambda_1 + \epsilon) = \mathbf{b} - \mathbf{b}^{-1} \mathbf{S} \cdot \epsilon \rho, \\ \text{and therefore } \mathbf{T}(\lambda_1 - \epsilon) + \mathbf{T}(\lambda_1 + \epsilon) &= 2\mathbf{b}; \end{aligned}$$

the locus of the extremity of the derived vector λ_1 is a certain *ellipsoid of revolution*, with the *mean axis* $2\mathbf{b}$ of the given ellipsoid for its *major axis*, and with *two foci* on that axis of which the vectors are $\pm \epsilon$; if e denote the *linear excentricity* of this *new ellipsoid*, $e = \mathbf{T}\epsilon$, then

$$e^2 = (a^2 - b^2)(b^2 - c^2)(a^2 - b^2 + c^2)^{-1};$$

the four vectors, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ terminate at four points, L_1, L_2, L_3, L_4 , which are the *four corners of a quadrilateral, inscribed in a circle, of this derived ellipsoid of revolution*; the two opposite sides, $L_1 L_2, L_3 L_4$, of this plane quadrilateral, are respectively *parallel to the two umbilical diameters* of the original ellipsoid abc ; the *two other* and mutually opposite sides, $L_2 L_3, L_4 L_1$, of the same inscribed quadrilateral, are parallel to the *axes of the two cylinders of revolution* which can be circumscribed about the same given ellipsoid (or to the asymptotes of the focal hyperbola); the former *pair of sides* of the inscribed but varying quadrilateral *intersect in a point* E (the termination of the vector ρ), of which *the locus is the given ellipsoid*; for this and for other reasons it is proposed to name the new ellipsoid of revolution the **MEAN ELLIPSOID**, and its foci the **TWO MEDIAL FOCI** of the given ellipsoid abc , Articles 680 to 688; Pages 688 to 700.

§ CXVII.* Many other geometrical applications may be made, of the same general principles; for example, if τ be a vector tangential to a line of curvature, then, with the significations of ι, κ, ν in §§ LXXVIII., LXXIX., we have the equations,

$$\mathbf{S} \cdot \nu \tau = 0, \quad \mathbf{S} \cdot \nu \tau \iota \kappa = 0, \quad \text{giving } \tau = \mathbf{UV} \cdot \nu \iota \mp \mathbf{UV} \cdot \nu \kappa;$$

this reproduces the known theorem, that *the lines of curvature on an ellipsoid bisect at each point the angles between the circular sections*; quaternions may also be employed to prove some theorems elsewhere published by the present writer, respecting the *curvature of a spherical conic*, . . .

Article 689; Page 700.

APPENDIX A (referred to in § CXIII.), Pages 701 to 716.

APPENDIX B (respecting the results of § CXIV.), Pages 717 to 730.

APPENDIX C (containing some additional account of the analysis by which some of those results were obtained), Pages 731 to the end.

[* The foregoing Analysis of the work into *Sections* did not occur to the author until it was too late to be incorporated with the text: but it has been printed here, as seeming likely to be useful.]

REFERENCES TO THE FIGURES.

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