ON SYMBOLICAL GEOMETRY

By

William Rowan Hamilton

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NOTE ON THE TEXT

The paper *On Symbolical Geometry*, by Sir William Rowan Hamilton, appeared in installments in volumes i–iv of *The Cambridge and Dublin Mathematical Journal*, for the years 1846–1849.

The articles of the paper appeared as follows:

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articles 40, 41	vol. iv (1849), pp. 84–89,
articles 42–53	vol. iv (1849), pp. 105–118.

Various errata noted by Hamilton have been corrected. In addition, the following obvious corrections have been made:—

in article 13, the sentence before equation (103), the word 'considered' has been changed to 'considering';

a missing comma has been inserted in equation (172);

in equation (209), '= 0' has been added;

in article 31, the sentence after equation (227), d' has been corrected to d'';

in equation (233), $\mathrm{a}_{\backslash}$ has been replaced in the second identity by $\mathrm{a}_{\backslash\backslash}$;

in equation (294), a superscript ^{2} has been applied to the the scalar term in the identity.

In the original publication, points in space are usually denoted with normal size capital roman letters (A, B, C etc.), but with 'small capitals' (A, B, C etc.) towards the end of the paper. In this edition, the latter typeface has been used throughout to denote points of space.

In the original publication, the operations of taking the vector and scalar part are usually denoted with capital roman letters V and S, but are on occasion printed in italic type. In this edition, these operations have been denoted by roman letters throughout.

David R. Wilkins
Dublin, June 1999

ON SYMBOLICAL GEOMETRY.

By Sir William Rowan Hamilton, LL.D., Dub. and Camb., P.R.I.A., Corresponding Member of the Institute of France and of several Scientific Societies, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.

INTRODUCTORY REMARKS.

The present paper is an attempt towards constructing a symbolical geometry, analogous in several important respects to what is known as symbolical algebra, but not identical therewith; since it starts from other suggestions, and employs, in many cases, other rules of combination of symbols. One object aimed at by the writer has been (he confesses) to illustrate, and to exhibit under a new point of view, his own theory, which has in part been elsewhere published, of algebraic quaternions. Another object, which interests even him much more, and will probably be regarded by the readers of this Journal as being much less unimportant, has been to furnish some new materials towards judging of the general applicability and usefulness of some of those principles respecting symbolical language which have been put forward in modern times. In connexion with this latter object he would gladly receive from his readers some indulgence, while offering the few following remarks.

An opinion has been formerly published* by the writer of the present paper, that it is possible to regard Algebra as a *science*, (or more precisely speaking) as a *contemplation*, in some degree *analogous to Geometry*, although not to be confounded therewith; and to separate it, as such, in our conception, from its own *rules of art* and *systems of expression*: and that when so regarded, and so separated, its ultimate subject-matter is found in what a great metaphysician has called the inner intuition of *time*. On which account, the writer ventured to characterise Algebra as being the *Science of Pure Time*; a phrase which he also expanded into this other: that it is (ultimately) the Science of *Order and Progression*. Without having as yet seen cause to abandon that former view, however obscurely expressed and imperfectly developed it may have been, he hopes that he has since profited by a study, frequently resumed, of some of the works of Professor Ohm, Dr. Peacock, Mr. Gregory, and some other authors; and imagines that he has come to seize their meaning, and appreciate their value, more fully than he was prepared to do, at the date of that former publication of his own to which he has referred. The whole theory of the laws and logic of symbols is indeed one of no small subtlety; insomuch that (as is well known to the readers of the *Cambridge Mathematical Journal*, in which periodical many papers of great interest and importance on this very subject have appeared) it requires a close and long-continued attention, in order to be able to form a judgment of any value respecting it: nor does the present writer venture to regard his own opinions on this head as being by any means sufficiently matured; much less does he desire to provoke a controversy with any of those who may perceive that he has

* *Trans. Royal Irish Acad.*, vol. XVII. Dublin, 1835.

not yet been able to adopt, in all respects, their views. That he has adopted *some* of the views of the authors above referred to, though in a way which does not seem to himself to be contradictory to the results of his former reflexions; and especially that he feels himself to be under important obligations to the works of Dr. Peacock upon Symbolical Algebra, are things which he desires to record, or mark, in some degree, by the very *title* of the present communication; in the course of which there will occur opportunities for acknowledging part of what he owes to other works, particularly to Mr. Warren's Treatise on the Geometrical Representation of the Square Roots of Negative Quantities.

Observatory of Trinity College, Dublin, Oct. 16, 1845.

Uniliteral and Biliteral Symbols.

1. In the following pages of an attempt towards constructing a symbolical geometry, it is proposed to employ (as usual) the roman capital letters A, B, &c., with or without accents, as symbols of *points* in space; and to make use (at first) of binary combinations of those letters, as symbols of straight *lines*: the symbol of the beginning of the line being written (for the sake of some analogies*) towards the right hand, and the symbol of the end towards the left. Thus BA will denote the line *to* B *from* A; and is not to be confounded with the symbol AB, which denotes a line having indeed the same extremities, but drawn in the opposite direction. A biliteral symbol, of which the two component letters denote determined and different points, will thus denote a finite straight line, having a determined length, direction, and situation in space. But a biliteral symbol of the particular form AA may be said to be a *null* line, regarded as the limit to which a line tends, when its extremities tend to coincide: the conception or at least the name and symbol of such a line being required for symbolic generality. All lines BA which are not null, may be called by contrast *actual*; and the two lines AB and BA may be said to be the *opposites* of each other. It will then follow that a null line is its own opposite, but that the opposites of two actual lines are always to be distinguished from each other.

On the mark =.

2. An *equation* such as

$$\mathrm{B} = \mathrm{A} \qquad (1),$$

between two uniliteral symbols, may be interpreted as denoting that A and B are *two names for one common point*; or that a point B, determined by one geometrical process, coincides with a point A determined by another process. When a formula of the kind (1) holds good, in any calculation, it is allowed to *substitute*, in any other part of that calculation, either of the two equated symbols for the other; and every other equation between two symbols of one common class must be interpreted as to allow a similar substitution. We shall not violate

* The writer regards the line to B from A as being in some sense an interpretation or construction of the symbol $\mathrm{B} - \mathrm{A}$; and the evident possibility of reaching the point B, by going along that line from the point A, may, as he thinks, be symbolized by the formula $\mathrm{B} - \mathrm{A} + \mathrm{A} = \mathrm{B}$.

this principle of symbolical language by interpreting as we shall interpret, an equation such as

$$\text{DC} = \text{BA} \tag{2},$$

between two biliteral symbols, as denoting that the two lines,* of which the symbols are equated, have *equal lengths and similar directions*, though they may have different situations in space: for if we call such lines *symbolically equal*, it will be allowed, in *this* sense of equality, which has indeed been already proposed by Mr. Warren, Dr. Peacock, and probably by some of the foreign writers referred to in Dr. Peacock's Report, as well as in that narrower sense which relates to magnitudes only, and for lines in space as well as those which are in one plane, to assert that lines *equal* to the same line are equal to each other. (Compare *Euclid*, XI. 9.) It will also be true, that

$$\text{D} = \text{B}, \quad \text{if DA} = \text{BA} \tag{3},$$

or in words, that the ends of two symbolically equal lines coincide if the beginnings do so; a consequence which it is very desirable and almost necessary that we should be able to draw, for the purposes of symbolical geometry, but which would not have followed, if an equation of the form (2) had not been interpreted so as to denote *only* equality of lengths, or *only* similarity of directions. The opposites of equal lines are equal in the sense above explained; therefore the equation (2) gives also this *inverse* equation,

$$\text{CD} = \text{AB} \tag{4}.$$

Lines joining the similar extremities of symbolically equal lines are themselves symbolically equal (*Euc.* I. 33); therefore the equation (2) gives also this *alternate* equation,

$$\text{DB} = \text{CA} \tag{5}.$$

The *identity* BA = BA gives, as its alternate equation,

$$\text{BB} = \text{AA} \tag{6},$$

which symbolic result may be expressed in words by saying that any two null lines are to be regarded as equal to each other. Lines equal to opposite lines may be said to be themselves opposite lines.

* The writer regards the relation between the two lines, mentioned in the text, as a sort of interpretation of the following symbolic equation, D − C = B − A; which may also denote that the point D is ordinally related (in space) to the point C as B is to A, and may in that view be also expressed by writing the *ordinal analogy*, D .. C :: B .. A; which admits of *inversion* and *alternation*. The same relation between four points may, as he thinks, be thus symbolically expressed D = B − A + C. But by writing it as an equation between lines, he deviates less from received notation.

On the mark +.

3. The equation*

$$\text{CB} + \text{BA} = \text{CA} \tag{7}$$

is true in the most elementary sense of the notation, when B is any point upon the finite straight line CA; but we propose now to *remove this restriction for the purposes of symbolical geometry*, and to regard the formula (7) as being universally *valid, by definition, whatever three points of space may be denoted* by the three letters ABC. The equation (7) will then *express nothing about those points*, but will serve to *fix the interpretation of the mark + when inserted between any two symbols of lines*; for if we meet any symbol formed by such insertion, suppose the symbol HG + FE, we have only to draw, or conceive drawn, from any assumed point A, a line BA = FE, and from the end B of the line so drawn, a new line CB = HG; and then the proposed symbol HG + FE will be interpreted by (7) as denoting the line CA, or at least a line equal thereto. In like manner, by defining that

$$\text{DC} + \text{CB} + \text{BA} = \text{DA} \qquad (8),$$

we shall be able to interpret any symbol of the form

$$\text{KI} + \text{HG} + \text{FE},$$

as denoting a determined (actual or null) line; at least if we now regard a line as *determined* when it is *equal* to a determined line: and similarly for any number of biliteral symbols, connected by marks + interposed. Calling *this* act of connection of symbols, the operation of *addition*; the added symbols, *summands*; and the resultant symbol, a *sum*; we may therefore now say, that the sum of any number of symbols of given lines is itself a symbol of a determined line; and that this symbolic sum of lines represents the *total* (or final) *effect* of all those successive rectilinear *motions*, or translations of a point in space, which are represented by the several summands. This *interpretation of a symbolic sum of lines* agrees with the conclusions already published by the authors above alluded to; though the methods of symbolically obtaining and expressing it, here given, may possibly be found to be new. The same interpretation satisfies, as it ought to do, the condition that the sums of equals shall be equal (compare the demonstration of *Euclid*, XI. 10); and also this other condition, almost as much required for the advantageous employment of symbolical language, that those lines which, when added to equal lines, give equal sums, shall be themselves equal lines: or that

$$\text{FE} = \text{DC}, \quad \text{if, } \text{FE} + \text{BA} = \text{DC} + \text{BA} \qquad (9).$$

* On the plan mentioned in former notes, this equation would be written as follows:

$$(\text{C} - \text{B}) + (\text{B} - \text{A}) = \text{C} - \text{A}.$$

It might be thus expressed: the ordinal relation of the point C to the point A is compounded of the relations of C to B and of B to A.

It shews too that the sum of two opposite lines, and generally that the sum of all the successive sides of any closed polygon, or of lines respectively equal to those sides, is a null line: thus

$$\text{AA} = \text{AB} + \text{BA} = \text{AC} + \text{CB} + \text{BA} = \&\text{c.} \qquad (10).$$

The symbolic sum of any two lines is found to be *independent of their order*, in virtue of the same interpretation; so that the equation

$$\text{FE} + \text{HG} = \text{HG} + \text{FE} \qquad (11),$$

is true, in the present system, *not as an independent definition*, but rather as one of the modes of *symbolically expressing that elementary theorem of geometry*, (*Euclid*, I. 33), on which was founded the rule for deducing, from any equation (2) between lines, the *alternate* equation (5). For if we assume, as we may, that three points A, B, C, have been so chosen as to satisfy the equations FE = BA, HG = CA; and that a fourth point D is chosen so as to satisfy the equation DC = BA; the same points will then, by the theorem just referred to, satisfy also the equation DB = CA; and the truth of the formula (11) will be proved, by observing that each of the two symbols which are equated in that formula is equal to the symbol DA, in virtue of the definition (7) of +, without any new definition: since

$$\text{FE} + \text{HG} = \text{DC} + \text{CA} = \text{DA} = \text{DB} + \text{BA} = \text{HG} + \text{FE}.$$

A like result is easily shown to hold good, for any number of summands; thus

$$\text{FE} + \text{HG} + \text{KI} = \text{KI} + \text{HG} + \text{FE} \qquad (12);$$

since the first member of this last equation may be put successively under the forms

$$(\text{FE} + \text{HG}) + \text{KI}, \quad \text{KI} + (\text{FE} + \text{HG}), \quad \text{KI} + (\text{HG} + \text{FE}),$$

and finally under the form of the second member; the stages of this successive transformations of symbols admitting easily of geometrical interpretations: and similarly in other cases. *Addition of lines in space* is therefore generally (as Mr. Warren has shewn it to be for lines in a single plane) a *commutative operation*; in the sense that the summands may interchange their places, without the sum being changed. It is also an *associative* operation, in the sense that any number of successive summands may be associated into one group, and collected into one partial sum (denoted by enclosing these summands in parentheses); and that then this partial sum may be added, as a single summand, to the rest: thus

$$(\text{KI} + \text{HG}) + \text{FE} = \text{KI} + (\text{HG} + \text{FE}) = \text{KI} + \text{HG} + \text{FE} \qquad (13).$$

On the mark −.

4. The equation*

$$\text{CA} - \text{BA} = \text{CB} \qquad (14)$$

* On the plan mentioned in some former notes, this equation would take the form

$$(\text{C} - \text{A}) - (\text{B} - \text{A}) = \text{C} - \text{B}.$$

is true, in the most elementary sense of the notation, when B is on CA; but we may remove this restriction by a *definitional extension* of the formula (14), for the purposes of symbolical geometry, as has been done in the foregoing article with respect to the formula (7); and then the equation (14), so extended, will express *nothing about the points* A, B, C, but will serve to fix the *interpretation of any symbol*, such as $\text{KI} - \text{FE}$, formed by *inserting the mark* $-$ *between the symbols of any two lines*. This general meaning of the effect of the mark $-$, so inserted, is consistent with the particular interpretation which suggested the formula (14); it is also consistent with the usual symbolical opposition between the effects of $+$ and $-$; since the comparison of (14) with (7) gives the equations

$$(\text{CA} - \text{BA}) + \text{BA} = \text{CA} \qquad (15),$$

and

$$(\text{CB} + \text{BA}) - \text{BA} = \text{CB} \qquad (16),$$

either of which two equations, if regarded as a general formula, and combined with the formula (7), would include, reciprocally, the definition (14) of $-$, and might be substituted for it.

Symbolical *subtraction* of one line from another is thus equivalent to the *decomposition* of a given rectilinear *motion* (CA) into two others, of which one (BA) is given; or to the *addition of the opposite* (AB) of the line which was to be subtracted: so that we may write the symbolical equation

$$-\text{BA} = +\text{AB} \qquad (17),$$

because the second member of (14) may be changed by (7) to $\text{CA} + \text{AB}$. These conclusions respecting symbolical subtraction of lines, differ only in their notation, and in the manner of arriving at them, from the results of the authors already referred to, so far as the present writer is acquainted with them. In the present notation, when an isolated biliteral symbol is preceded with $+$ or $-$, we may still interpret it as denoting a line, if we agree to prefix to it, for the purpose of such interpretation, the symbol of a null line; thus we may write

$$+\text{AB} = \text{AA} + \text{AB} = \text{AB}, \quad -\text{AB} = \text{AA} - \text{AB} = \text{BA} \qquad (18);$$

$+$AB will, therefore, on this plan, be another symbol for the line AB itself, and $-$AB will be a symbol for the opposite line BA.

Abridged Symbols for Lines.

5. Some of the foregoing formulæ may be presented more concisely, and also in a way more resembling ordinary Algebra, by using now some new *uniliteral* symbols, such as the small roman letters a, b, &c., with or without accents, as symbols of lines, instead of binary combinations of the roman capitals, in cases where the lines which are compared are not supposed to have necessarily any common point, and generally when the *situations* of lines are disregarded, but not their lengths nor their directions. Thus we shall have, instead of (11) and (12), (13), (15) and (16), these other formulæ of the present Symbolical Geometry, which agree in all respect with those used in Symbolical Algebra:

$$\text{a} + \text{b} = \text{b} + \text{a}, \quad \text{a} + \text{b} + \text{c} = \text{c} + \text{b} + \text{a} \qquad (19);$$

$$(\mathrm{c}+\mathrm{b})+\mathrm{a}=\mathrm{c}+(\mathrm{b}+\mathrm{a})=\mathrm{c}+\mathrm{b}+\mathrm{a} \qquad (20);$$

$$(\mathrm{b}-\mathrm{a})+\mathrm{a}=\mathrm{b}, \quad (\mathrm{b}+\mathrm{a})-\mathrm{a}=\mathrm{b} \qquad (21);$$

and because the isolated but *affected* symbols +a, −a, may denote, by (18), the line a itself, and the opposite of that line, we have also here the usual *rule of the signs*,

$$+(+\mathrm{a})=-(-\mathrm{a})=+\mathrm{a}, \quad +(-\mathrm{a})=-(+\mathrm{a})=-\mathrm{a} \qquad (22).$$

Introduction of the marks × and ÷.

6. Continuing to denote lines by letters, the formula

$$(\mathrm{b}\div\mathrm{a})\times\mathrm{a}=\mathrm{b} \qquad (23),$$

which is, for the relation between multiplication and division, what the first of the two formulæ (21) is for the relation between addition and subtraction, will be true, in the most elementary sense of the multiplication of a length by a number, for the case when the line b is the sum of several summands, each equal to the line a, and when the number of those summands is denoted by the quotient b ÷ a. And we shall now, for the purposes of symbolical generality, *extend* this formula (23), so as to make it be valid, *by definition*, *whatever two lines* may be denoted by a and b. The formula will then *express nothing respecting those lines* themselves, which can serve to distinguish them from any other lines in space; but will furnish a *symbolic condition*, which we must satisfy by the *general interpretation* of a *geometrical quotient*, and of the *operation of multiplying a line* by such a quotient.

To make such a general interpretation consistent with the particular case where the quotient becomes a *quotity*, we are led to write

$$\mathrm{a}\div\mathrm{a}=1, \quad (\mathrm{a}+\mathrm{a})\div\mathrm{a}=2, \ \text{\&c.} \qquad (24),$$

and conversely

$$1\times\mathrm{a}=\mathrm{a}, \quad 2\times\mathrm{a}=\mathrm{a}+\mathrm{a} \ \text{\&c.} \qquad (25);$$

and because, when quotients can be thus interpreted as quotities, the four equations

$$(\mathrm{c}\div\mathrm{a})+(\mathrm{b}\div\mathrm{a})=(\mathrm{c}+\mathrm{b})\div\mathrm{a} \qquad (26),$$

$$(\mathrm{c}\div\mathrm{a})-(\mathrm{b}\div\mathrm{a})=(\mathrm{c}-\mathrm{b})\div\mathrm{a} \qquad (27),$$

$$(\mathrm{c}\div\mathrm{a})\times(\mathrm{a}\div\mathrm{b})=\mathrm{c}\div\mathrm{b} \qquad (28),$$

$$(\mathrm{c}\div\mathrm{a})\div(\mathrm{b}\div\mathrm{a})=\mathrm{c}\div\mathrm{b} \qquad (29),$$

are true in the most elementary sense of arithmetical operations on whole numbers, we shall now *define* that these four equations are valid, *whatever three lines* may be denoted by a, b, c; and thus shall have conditions for the general *interpretations of the four operations* + − × ÷ *performed on geometrical quotients.*

We shall in this way be led to interpret a quotient of which the divisor is an actual line, but the dividend a null one, as being equivalent to the symbol $1 - 1$ or *zero*; so that

$$(\mathrm{a} - \mathrm{a}) \div \mathrm{a} = 0, \quad 0 \times \mathrm{a} = \mathrm{a} - \mathrm{a} \qquad (30).$$

Negative numbers will present themselves in the consideration of such quotients and products as

$$(-\mathrm{a}) \div \mathrm{a} = 0 - 1 = -1, \quad (-1) \times \mathrm{a} = -\mathrm{a}, \text{ \&c.} \qquad (31);$$

fractional numbers in such formulæ as

$$\mathrm{a} \div (\mathrm{a} + \mathrm{a}) = 1 \div 2 = \tfrac{1}{2}, \quad \tfrac{1}{2} \times (\mathrm{a} + \mathrm{a}) = \mathrm{a}, \text{ \&c.} \qquad (32);$$

and *incommensurable* numbers, by the conception of the connected *limits* of quotients and products, and by the formula, which symbolical language leads us to assume,

$$\left(\lim \frac{n}{m}\right) \times \mathrm{a} = \lim \left(\frac{n}{m} \times \mathrm{a}\right) \qquad (33).$$

If then we give the name of SCALARS to all numbers of the kind called usually *real*, because they are all contained on the one *scale* of progression of number from negative to positive infinity; and if we agree, for the present, to denote such numbers generally by small italic letters a, b, c, &c.; and to insert the mark $\parallel$ between the symbols of two lines when we wish to express that the directions of those lines are either exactly similar or exactly opposite to each other, in each of which two cases the lines may be said to be *symbolically parallel*; we shall have generally two equations of the forms

$$\mathrm{b} \div \mathrm{a} = a, \quad a \times \mathrm{a} = \mathrm{b}, \quad \text{when } \mathrm{b} \parallel \mathrm{a} \qquad (34).$$

That is to say, the *quotient of two parallel lines* is generally a *scalar number*; and, conversely, to multiply a given line (a) by a given scalar (or real) number a, is to determine a new line (b) parallel to the given line (a), the direction of the one being similar or opposite to that of the other, according as the number is positive or negative, while the length of the new line bears to the length of the given line a ratio which is marked by the same given number. So that if A_0 A_1 A_a denote any three points on one common axis of rectilinear progression, which are related to each other, upon that axis, as to their order and their intervals, in the same manner as the three scalar numbers 0, 1, a, regarded as ordinals, are related to each other on the scale of numerical progression from $-\infty$ to $+\infty$, then the equations

$$\mathrm{A}_a\mathrm{A}_0 \div \mathrm{A}_1\mathrm{A}_0 = a, \quad a \times \mathrm{A}_1\mathrm{A}_0 = \mathrm{A}_a\mathrm{A}_0 \qquad (35)$$

will be true by the foregoing interpretations.

It is easy to see that this mode of interpreting a quotient of parallel lines renders the formulæ (26) (27) (28) (29) consistent with the received rules for performing the operations $+ - \times \div$ on what are called the real numbers, whether they be positive or negative, and whether commensurable or incommensurable; or rather reproduces those rules as consequences of those formulæ.

7. The other chief relation of directions of lines in space, besides parallelism, is perpendicularity; which it is not unusual to denote by writing the mark $\perp$ between the symbols of two perpendicular lines. And the other chief class of geometrical quotients which it is important to study, as preparatory to a general theory of such quotients, is the class in which the dividend is a line perpendicular to the divisor. A quotient of this latter class we shall call a VECTOR, to mark its connection (which is closer than that of a *scalar*) with the conception of *space*, and for other reasons which will afterwards appear: and if we agree to denote, for the present, such vector quotients (of perpendicular lines) by small Greek letters, in contrast to the scalar class of quotients (of parallel lines) which we have proposed to denote by small italic letters, we shall then have generally two equations of the forms

$$\mathrm{c} \div \mathrm{a} = \alpha, \quad \mathrm{c} = \alpha \times \mathrm{a}, \quad \text{if } \mathrm{c} \perp \mathrm{a} \qquad (36).$$

Any line e may be put under the form c + b, in which b $\parallel$ a, and c $\perp$ a; a *general geometrical quotient* may therefore, by (26) (34) (36), be considered as the *symbolic sum of a scalar and a vector*, zero being regarded as a common limit of quotients of these two classes; and consequently, if we adopt the notation just now mentioned, we have generally an equation of the form

$$\mathrm{e} \div \mathrm{a} = \alpha + a \qquad (37).$$

This *separation of the scalar and vector parts* of a general geometrical quotient corresponds (as we see) to the decomposition, by *two separate projections*, of the dividend line into two other lines of which it is the symbolic sum, and of which one is parallel to the divisor line, while the other is perpendicular thereto. To be able to mark on some occasions more distinctly, in writing, than by the use of two different alphabets, the conception of such separation, we shall here introduce two new symbols of operation, namely the abridged words Scal. and Vect., which, where no confusion seems likely to arise from such farther abridgment, we shall also denote more shortly still by the letters S and V, prefixing them to the symbol of a general geometrical quotient in order to form separate symbols of its scalar and vector parts; so that we shall now write more generally, for any two lines a and e,

$$\mathrm{e} \div \mathrm{a} = \mathrm{Vect.}(\mathrm{e} \div \mathrm{a}) + \mathrm{Scal.}(\mathrm{e} \div \mathrm{a}) \qquad (38);$$

or more concisely,

$$\mathrm{e} \div \mathrm{a} = \mathrm{V}(\mathrm{e} \div \mathrm{a}) + \mathrm{S}(\mathrm{e} \div \mathrm{a}) \qquad (39);$$

in which expression the order of the two summands may be changed, in virtue of the definition (26) of addition of geometrical quotients, because the order of the two partial dividends may be changed without preventing the dividend line e from being still their symbolic sum. A scalar cannot become equal to a vector, except by each becoming zero; for if the divisor of the vector quotient be multiplied separately by the scalar and the vector, the products of these two multiplications will be (by what has been already shown) respectively lines parallel and perpendicular to that divisor, and therefore not symbolically equal to each other, except it be at the limit where both become null lines, and are on that account regarded as equal. A scalar quotient $\mathrm{b} \div \mathrm{a} = a$, $(\mathrm{b} \parallel \mathrm{a})$, has been seen to denote the relative length and relative direction (as similar or opposite) of two parallel lines a, b: and in like manner a vector quotient $\mathrm{e} \div \mathrm{a} = \alpha$, $(\mathrm{c} \perp \mathrm{a})$, may be regarded as denoting the *relative length and relative direction* (depending on *plane* and *hand*) *of two perpendicular lines* a, c; or as indicating at once *in what ratio* the length of one line a must be altered (if at all) in order to become equal to the length of another line c, and also *round what axis*, perpendicular to both these two rectangular lines, the direction of the divisor line a must be caused or conceived to turn, right-handedly, through a right angle, in order to attain the original direction of the dividend line c. A line drawn in the direction of this *axis of* (what is here regarded as) *positive rotation*, and having its length in the same ratio to some assumed *unit* of length as the length of the dividend to that of the divisor, may be called the INDEX of the vector. We shall thus be led to substitute, for any equation between two vector quotients, an equation between two lines, namely between their indices; for if we define that two vector quotients, such as $\mathrm{c} \div \mathrm{a}$ and $\mathrm{c}' \div \mathrm{a}'$ if $\mathrm{c} \perp \mathrm{a}$ and $\mathrm{c}' \perp \mathrm{a}'$, are *equal* when they have *equal indices*, we shall satisfy all conditions of symbolical equality, of the kinds already considered in connection with other definitions; we shall also be able to say that in every case of two such equal quotients, the two dividend lines (c and c′) bear to their own divisor lines (a and a′), respectively, one common ratio of lengths, and one common relation of directions. We shall thus also, by (23), be able to *interpret the multiplication* of any given line a′ by any given vector $\mathrm{c} \div \mathrm{a}$, *provided that the one is perpendicular to the index of the other*, as the operation of deducing from a′ another line c′, by altering (generally) its length in a given ratio, and by turning (always) its direction round a given axis of rotation, namely round the index of the vector, right-handedly, through a right angle. And we can now *interpret an equation between two general geometrical quotients*, such as

$$\mathrm{e}' \div \mathrm{a}' = \mathrm{e} \div \mathrm{a} \qquad (40),$$

as being equivalent to a *system of two separate equations*, one between the scalar and another between the vector parts, namely the two following:

$$\mathrm{S}(\mathrm{e}' \div \mathrm{a}') = \mathrm{S}(\mathrm{e} \div \mathrm{a}); \quad \mathrm{V}(\mathrm{e}' \div \mathrm{a}') = \mathrm{V}(\mathrm{e} \div \mathrm{a}) \qquad (41);$$

of which each separately is to be interpreted on the principles already laid down; and which are easily seen (by considerations of similar triangles) to imply, when taken jointly, that the length of e′ is to that of a′ in the same ratio as the length of e to that of a; and also that the same rotation, round the index of either of the two equal vectors, which would cause the direction of a to attain the original direction of e, would also bring the direction of a′ into that originally occupied by e′. At the same time we see how to interpret the operation of multiplying any given line a′ by any given geometrical quotient $\mathrm{e} \div \mathrm{a}$ of two other lines, *whenever the three given lines* a*,* e*,* a′*, are parallel to one common plane*; namely as being the complex operation of altering (generally) a given length in a given ratio, and of turning a given line round a given axis, through a given amount of right-handed rotation, in order to obtain a certain new line e′, which may be thus denoted, in conformity with the definition (23),

$$\mathrm{e}' = (\mathrm{e} \div \mathrm{a}) \times \mathrm{a}' \qquad (42).$$

The relation between the four lines a, e, a′, e′, may also be called a *symbolic analogy*, and may be thus denoted:

$$\mathrm{e}' : \mathrm{a}' :: \mathrm{e} : \mathrm{a} \qquad (43);$$

a' and e being the *means*, and e' and a the *extremes* of the analogy. An analogy or equation of this sort admits (as it is easy to prove) of *inversion* and *alternation*; thus (43) or (42) gives, *inversely*,

$$a' : e' :: a : e, \quad a' \div e' = a \div e \tag{44},$$

and *alternately*,

$$e' : e :: a' : a, \quad e' \div e = a' \div a \tag{45}.$$

These results respecting analogies between *co-planar lines*, that is, between lines which are in or parallel to one common plane, agree with, and were suggested by, the results of Mr. Warren. But it will be necessary to introduce other principles, or at least to pursue farther the track already entered on, before we can arrive at an interpretation of a *fourth proportional to three lines which are not parallel to any common plane*: or can interpret the multiplication of a line by a quotient of two others, when it is not perpendicular to what has been lately called the index of the vector part of that quotient.

Determinateness of the first Four Operations on Geometrical Fractions (or Quotients).

8. Meanwhile the principles and definitions which have been already laid down, are sufficient to conduct to clear and determinate interpretations of all operations of combining geometrical quotients among themselves, by any number of additions, subtractions, multiplications, and divisions: each *quotient* of the kind here mentioned being regarded, by what has been already shown, as the *mark of a certain complex relation between two straight lines in space*, depending not only on their *relative lengths*, but also on their *relative directions*. If we denote now by a symbol of fractional form, such as $\frac{b}{a}$, the quotient thus obtained by dividing one line b by another line a, when directions as well as lengths are attended to, the definitional equations (26), (27), (28), (29), will take these somewhat shorter forms:*

$$\frac{c}{a} + \frac{b}{a} = \frac{c+b}{a}; \quad \frac{c}{a} - \frac{b}{a} = \frac{c-b}{a}; \tag{46},$$

* On the principles alluded to in former notes, the formulæ for the addition, subtraction, multiplication, and division, of any two geometrical fractions, might be thus written:

$$\frac{D-C}{B-A} + \frac{C-A}{B-A} = \frac{D-A}{B-A},$$
$$\frac{D-A}{B-A} - \frac{C-A}{B-A} = \frac{D-C}{B-A},$$
$$\frac{D-A}{C-A} \times \frac{C-A}{B-A} = \frac{D-A}{B-A},$$
$$\frac{D-A}{B-A} \div \frac{C-A}{B-A} = \frac{D-A}{C-A};$$

A, B, C, D being symbols of any four points of space, and $B - A$ being a symbol of the straight line drawn to B from A. If we denote this line by the biliteral symbol BA, we obtain the following somewhat shorter forms, which do not however all agree so closely with the forms

$$\frac{c}{a} \times \frac{a}{b} = \frac{c}{b}; \quad \frac{c}{a} \div \frac{b}{a} = \frac{c}{b}; \tag{47}$$

which agree in all respects with the corresponding formulæ of ordinary algebra, and serve to fix, in the present system, the meanings of the operations $+$, $-$, $\times$, $\div$, on what may be called *geometrical fractions.* These FRACTIONS being only other forms for what we have called *geometrical quotients* in earlier articles of ths paper, we may now write the identity,

$$\frac{b}{a} = b \div a \tag{48}$$

For the same reason, an *equation between any two such fractions*, for example the following,

$$\frac{f}{e} = \frac{b}{a} \tag{49}$$

is to be understood as signifying, 1st, that the *length* of the one *numerator* line f is to the length of its own *denominator* line e *in the same ratio* as the length of the other numerator line b to the length of the other denominator line a; 2nd, that these four lines are *co-planar*, that is to say, in or parallel to one common plane; and 3rd, that the *same amount and direction of rotation*, round an axis perpendicular to this common plane, which would bring the line a into the direction originally occupied by b, would also bring the line e into the original direction of f. The same complex relation between the same four lines may also (by what has been already seen) be expressed by the *inverse* equation

$$\frac{e}{f} = \frac{a}{b} \tag{50}$$

or by the *alternate* form

$$\frac{f}{b} = \frac{e}{a} \tag{51}$$

Two fractions which are, in this sense, *equal* to the same third fraction, are also equal to each other; and the *value* of such a fraction is not altered by altering the lengths of its numerator and denominator in any common ratio; nor by causing both to turn together through any common amount of rotation, in a common direction, round an axis perpendicular to both; nor by transporting either or both, without rotation, to any other positions in space. When

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of ordinary algebra:

$$\frac{DC}{BA} + \frac{CA}{BA} = \frac{DA}{BA},$$
$$\frac{DA}{BA} - \frac{CA}{BA} = \frac{DC}{BA},$$
$$\frac{DA}{CA} \times \frac{CA}{BA} = \frac{DA}{BA},$$
$$\frac{DA}{BA} \div \frac{CA}{BA} = \frac{DA}{CA}.$$

the lengths and directions of any three co-planar lines, a, b, e, are given, it is always possible to determine the length and direction of a fourth line f, which shall be co-planar with them, and shall satisfy an equation between fractions, of the form (49). It is therefore possible to *reduce any two geometrical fractions to a common denominator*; or to satisfy not only the equation (49), but also this other equation,

$$\frac{\mathrm{h}}{\mathrm{g}} = \frac{\mathrm{c}}{\mathrm{a}} \tag{52}$$

by a suitable choice of the three lines a, b, c, when the four lines e, f, g, h, are given; since, whatever may be the given directions of these four lines, it is always possible to find (or to conceive as found) a fifth line a, which shall be at once co-planar with the pair e, f and also with the pair g, h. For a similar reason it is always possible to transform two given geometrical fractions into two others equivalent to them, in such a manner, that the new denominator of one shall be equal to the new numerator of the other; or to satisfy the two equations

$$\frac{\mathrm{h}}{\mathrm{g}} = \frac{\mathrm{c}'}{\mathrm{a}'}, \quad \frac{\mathrm{f}}{\mathrm{e}} = \frac{\mathrm{a}'}{\mathrm{b}'} \tag{53}$$

by a suitable choice of the three lines a′, b′, c′, whatever the four given lines e, f, g, h may be. Making then for abridgment

$$\mathrm{c} + \mathrm{b} = \mathrm{d}, \quad \mathrm{c} - \mathrm{b} = \mathrm{d}' \tag{54}$$

and interpreting a sum or difference of lines as has been done in former articles, we see that it is always possible to choose eight lines a, b, c, d, a′, b′, c′, d′, so as to satisfy the conditions (49), (52), (53), (54); and thus, by (46) and (47), to interpret the sum, the difference, the product, and the quotient of *any two* given geometrical fractions, $\frac{\mathrm{f}}{\mathrm{e}}$ and $\frac{\mathrm{h}}{\mathrm{g}}$, as being each equal to *another given fraction* of the same sort, as follows:

$$\frac{\mathrm{h}}{\mathrm{g}} + \frac{\mathrm{f}}{\mathrm{e}} = \frac{\mathrm{d}}{\mathrm{a}}, \quad \frac{\mathrm{h}}{\mathrm{g}} - \frac{\mathrm{f}}{\mathrm{e}} = \frac{\mathrm{d}'}{\mathrm{a}} \tag{55}$$

$$\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}} = \frac{\mathrm{c}'}{\mathrm{b}'}, \quad \frac{\mathrm{h}}{\mathrm{g}} \div \frac{\mathrm{f}}{\mathrm{e}} = \frac{\mathrm{c}}{\mathrm{b}} \tag{56}$$

any variations in the new numerators and denominators, which are consistent with the foregoing conditions, being easily seen to make no changes in the values of the fractions which result. The *interpretations* of these four symbolic combinations, which are the first members of the four equations (55) and (56), are thus entirely *fixed*: and we are *no longer at liberty, in the present system*, to introduce arbitrarily any *new meanings* for those symbolical forms, or to subject them to any *new laws* of combination among themselves, without examining whether such meanings or such laws are consistent with the principles and definitions which it has been thought right to establish already, as appearing to be more simple and primitive, and more intimately connected with the application of symbolical language to geometry, or at least with the plan on which it is here attempted to make that application, than any of those other laws or meanings. If, for example, it shall be found that, in virtue of the foregoing principles, the *successive addition* of any number of geometrical fractions gives a result which is independent of their order, this consequence will be, for us, a *theorem*, and not a definition. And if, on the contrary, the same principles shall lead us to regard the *multiplication* of geometrical fractions as being in general a *non-commutative* operation, or as giving a result which is *not* independent of the order of the factors, we shall be obliged to accept this conclusion also, that we may preserve consistency of system.

Separation of the Scalar and Vector parts of Sums and Differences of Geometrical Fractions.

9. To develope the geometrical meaning of the first equation (46), we may conceive each of the two numerator lines b, c, and also their sum d, to be orthogonally projected, first on the common denominator line a itself, and secondly on a plane perpendicular to that denominator. The former projections may be called b_1, c_1, d_1; the latter b_2, c_2, d_2; and thus we shall have the nine relations,

$$\left.\begin{array}{lll} b_2 + b_1 = b, & b_1 \parallel a, & b_2 \perp a, \\ c_2 + c_1 = c, & c_1 \parallel a, & c_2 \perp a, \\ d_2 + d_1 = d, & d_1 \parallel a, & d_2 \perp a, \end{array}\right\} \qquad (57),$$

together with the three equations

$$c + b = d, \quad c_1 + b_1 = d_1, \quad c_2 + b_2 = d_2 \qquad (58);$$

of which the last two are deducible from the first, by the geometrical properties of projections. We have, therefore, by (46),

$$\frac{c}{a} + \frac{b}{a} = \frac{d}{a} = \frac{d_2}{a} + \frac{d_1}{a} \qquad (59),$$

$$\frac{d_1}{a} = \frac{c_1}{a} + \frac{b_1}{a}, \quad \frac{d_2}{a} = \frac{c_2}{a} + \frac{b_2}{a} \qquad (60).$$

Since the three projections b_1, c_1, d_1, are parallel to a (in that sense of the word *parallel* which does not exclude coincidence), the three quotients in the first equation (60) are what we have already named *scalars*; that is, they are what are commonly called real numbers, positive, negative, or zero: they are also the scalar parts of the three quotients in the first equation (59), so that we may write

$$\frac{b_1}{a} = S\frac{b}{a}, \quad \frac{c_1}{a} = S\frac{c}{a}, \quad \frac{d_1}{a} = S\frac{d}{a} \qquad (61),$$

using the letter S here, as in a former article, for the characteristic of the operation of *taking the scalar part* of any geometrical quotient, or fraction. (If any confusion should be apprehended, on other occasions, from this use of the letter S, and if the abridged word Scal. should be thought too long, the sign $\underset{c}{S}$ might be employed.) Eliminating the four symbols

b_1, c_1, d_1, d, between the first equation (59), the first equation (60), and the three equations (61), we obtain the result

$$S\left(\frac{c}{a}+\frac{b}{a}\right)=S\frac{c}{a}+S\frac{b}{a} \qquad (62);$$

in which, by the foregoing article, $\frac{b}{a}$ and $\frac{c}{a}$ may represent any two geometrical fractions: so that we may write generally

$$S\left(\frac{h}{g}+\frac{f}{e}\right)=S\frac{h}{g}+S\frac{f}{e} \qquad (63),$$

and may enunciate in words the same result by saying, that the *scalar* of the sum of any two such fractions is equal to the *sum of the scalars.* In like manner, the three other projections b_2, c_2, d_2, being each perpendicular to a, the three other partial quotients, which enter into the second equation (60), are what we have already called *vectors* in this paper, or more fully they are the vector parts of the three quotients in the first equation (59); so that we may write

$$\frac{b_2}{a}=V\frac{b}{a},\quad \frac{c_2}{a}=V\frac{c}{a},\quad \frac{d_2}{a}=V\frac{d}{a} \qquad (64),$$

V being used, as in a former article, for the characteristic of the operation of *taking the vector part*; we have, therefore,

$$V\left(\frac{c}{a}+\frac{b}{a}\right)=V\frac{c}{a}+V\frac{b}{a} \qquad (65),$$

$$V\left(\frac{h}{g}+\frac{f}{e}\right)=V\frac{h}{g}+V\frac{f}{e} \qquad (66),$$

and may assert that the *vector of the sum* of any two geometrical fractions is equal to the *sum of the vectors.* These formulæ (63) and (66) are important in the present system; they are however, as we see, only symbolical expressions of those very simple geometrical principles from which they have been derived, through the medium of the equations (58); namely, the principles that, *whether on a line or on a plane*, the *projection of a sum* of lines is equal to the *sum of the projections*, if the word *sum* be suitably interpreted. The analogous interpretation of a *difference* of lines, combined with similar considerations, gives in like manner the formulæ

$$S\left(\frac{h}{g}-\frac{f}{e}\right)=S\frac{h}{g}-S\frac{f}{e} \qquad (67),$$

$$V\left(\frac{h}{g}-\frac{f}{e}\right)=V\frac{h}{g}-V\frac{f}{e} \qquad (68);$$

that is to say, the *scalar and vector of the difference* of any two geometrical fractions are respectively equal to the *differences of the scalars and of the vectors* of those fractions; precisely as, and because, the *projection of a difference* of two lines, whether on a line or on a plane, is equal to the *difference of the projections.*

10. We see, then, that in order to combine by addition or subtraction any two geometrical fractions, it is sufficient to combine separately their scalar and their vector parts. The former parts, namely the scalars, are simply *numbers*, of the kind called commonly real; and are to be added or subtracted among themselves according to the usual rules of algebra. But for effecting with convenience the combination of the latter parts among themselves, namely the vectors, which have been shown in a former article to be of a kind essentially distinct from all stages of the progression of real number from negative to positive infinity (and therefore to be rather *extra-positives* than either positive or *contra*-positive numbers), it is necessary to establish other rules: and it will be found useful for this purpose to employ the consideration of certain connected *lines*, namely the *indices*, of which each is determined by, and in its turn completely characterises, that vector quotient or fraction to which it corresponds, according to the construction assigned in the 7th article. If we apply the rules of that construction to determine the indices of the vector parts of any two fractions and of their sum, we may first, as in recent articles, reduce the two fractions to a common denominator; and may, for simplicity, take this denominator line a of a length equal to that assumed unit of length which is to be employed in the determination of the indices. Then, having projected, as in the last article, the new numerators b and c, and their sum d, on a plane perpendicular to a, and having called these projections b_2, c_2, d_2, as before; we may conceive a right-handed rotation of each of these three projected lines, through a right angle, round the line a as a common axis, which shall transport them without altering their lengths or relative directions, and therefore without affecting their mutual relation as summands and sum, into coincidence with three other lines b_3, c_3, d_3, such that

$$d_3 = c_3 + b_3 \qquad (69);$$

and these three new lines will be the three indices required. For a right-handed rotation through a right angle, round the line b_3 as an axis, would bring the line a into the direction originally occupied by b_2; and the length of b_2 is to the length of a in the same ratio as the length of b_3 to the assumed unit of length; therefore b_3 is, in the sense of the 7^{th} article, the index of the vector quotient $\frac{b_3}{a}$, that is, the index of the vector part of the fraction $\frac{b}{a}$, or $\frac{f}{e}$; and similarly for the indices of the two other fractions, in the first equation (59). We may therefore write, as consequences of the construction lately assigned, and of the equations (49) and (52),

$$b_3 = \mathrm{I}\frac{f}{e}; \quad c_3 = \mathrm{I}\frac{h}{g}; \quad d_3 = \mathrm{I}\left(\frac{f}{e} + \frac{h}{g}\right) \qquad (70);$$

if we agree for the present to prefix the letter I to the symbol of a geometrical fraction, as the characteristic of the operation of *taking the index of the vector part*. Eliminating now the three symbols b_3, c_3, d_3 between the four equations (69) and (70), we obtain this general formula:

$$\mathrm{I}\left(\frac{h}{g} + \frac{f}{e}\right) = \mathrm{I}\frac{h}{g} + \mathrm{I}\frac{f}{e} \qquad (71),$$

which may be thus enunciated: the *index of the vector part of the sum* of any two geometrical fractions is equal to the *sum of the indices* of the vector parts of the summands. Combining this result with the formula (63), which expresses that the scalar of the sum is the sum of the scalars, we see that the complex *operation of adding any two geometrical fractions*, of which each is determined by its scalar and by the index of its vector part, may be in general *decomposed into two* very simple but *essentially distinct operations*; namely, *first*, the operation of adding together *two numbers*, positive or negative or null, so as to obtain a third number for their sum, according to the usual rules of elementary algebra; and *second*, the operation of adding together *two lines* in space, so as to obtain a third line, according to the geometrical rules of the composition of motions, or by drawing the diagonal of a parallelogram. In like manner the operation of *taking the difference* of two fractions may be decomposed into the two operations of taking separately the difference of two numbers, and the difference of two lines; for we can easily prove that

$$\mathrm{I}\left(\frac{\mathrm{h}}{\mathrm{g}} - \frac{\mathrm{f}}{\mathrm{e}}\right) = \mathrm{I}\frac{\mathrm{h}}{\mathrm{g}} - \mathrm{I}\frac{\mathrm{f}}{\mathrm{e}} \qquad (72);$$

or, in words, that the *index* (of the vector part) *of the difference* of any two fractions is equal to the *difference of the indices*. And because it has been seen that not only for numbers but also for lines, considered amongst themselves, any number of summands may be in any manner grouped or transposed without altering the sum; and that the sum of a scalar and a vector is equal to the sum of the same vector and the same scalar, combined in a contrary order; it follows that the *addition* of any number of geometrical fractions is an *associative* and also a *commutative* operation: in such a manner that we may now write

$$\frac{\mathrm{h}}{\mathrm{g}} + \frac{\mathrm{f}}{\mathrm{e}} = \frac{\mathrm{f}}{\mathrm{e}} + \frac{\mathrm{h}}{\mathrm{g}}; \quad \frac{\mathrm{k}}{\mathrm{i}} + \left(\frac{\mathrm{h}}{\mathrm{g}} + \frac{\mathrm{f}}{\mathrm{e}}\right) = \left(\frac{\mathrm{k}}{\mathrm{i}} + \frac{\mathrm{h}}{\mathrm{g}}\right) + \frac{\mathrm{f}}{\mathrm{e}} = \frac{\mathrm{f}}{\mathrm{e}} + \frac{\mathrm{h}}{\mathrm{g}} + \frac{\mathrm{k}}{\mathrm{i}}, \ \&\mathrm{c.} \qquad (73),$$

whatever straight lines in space may be denoted by e, f, g, h, i, k, &c. We may also write, concisely

$$\mathrm{S}\textstyle\sum = \sum\mathrm{S}; \quad \mathrm{V}\sum = \sum\mathrm{V}; \quad \mathrm{I}\sum = \sum\mathrm{I} \qquad (74);$$

$$\mathrm{S}\Delta = \Delta\mathrm{S}; \quad \mathrm{V}\Delta = \Delta\mathrm{V}; \quad \mathrm{I}\Delta = \Delta\mathrm{I} \qquad (75);$$

using $\sum$, Δ as the characteristics of sum and difference, while S, V and I are still the signs of scalar, vector, index.

Separation of the Scalar and Vector Parts of the Product of any two Geometrical Fractions.

11. The definitions (46), (47) of addition and multiplication of fractions, namely

$$\frac{\mathrm{c}}{\mathrm{a}} + \frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{c}+\mathrm{b}}{\mathrm{a}}, \quad \frac{\mathrm{c}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{b}},$$

give obviously, for any 4 straight lines a, b, c, a′, the formula

$$\left(\frac{\mathrm{c}}{\mathrm{a}} + \frac{\mathrm{b}}{\mathrm{a}}\right) \times \frac{\mathrm{a}}{\mathrm{a}'} = \frac{\mathrm{c}+\mathrm{b}}{\mathrm{a}'} = \left(\frac{\mathrm{c}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{a}'}\right) + \left(\frac{\mathrm{b}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{a}'}\right) \qquad (76);$$

and this other formula of the same kind,

$$\frac{\mathrm{a}'}{\mathrm{a}} \times \left(\frac{\mathrm{c}}{\mathrm{a}} + \frac{\mathrm{b}}{\mathrm{a}}\right) = \frac{\mathrm{a}'}{\dfrac{\mathrm{a}}{\mathrm{c}+\mathrm{b}} \times \mathrm{a}} = \left(\frac{\mathrm{a}'}{\mathrm{a}} \times \frac{\mathrm{c}}{\mathrm{a}}\right) + \left(\frac{\mathrm{a}'}{\mathrm{a}} \times \frac{\mathrm{b}}{\mathrm{a}}\right) \qquad (77),$$

may be proved without difficulty to be a consequence of the same definitions; the operation of multiplying a line, by the quotient of two others with which it is co-planar, being interpreted by the definition (23), so as to give, in the present notation,

$$\frac{\mathrm{e}}{\mathrm{a}} \times \mathrm{a} = \mathrm{e} \qquad (78).$$

In fact, if we assume, as we may, seven new lines $\mathrm{d}\mathrm{b}'\mathrm{c}'\mathrm{d}'\mathrm{b}''\mathrm{c}''\mathrm{d}''$, so as to satisfy the seven conditions

$$\left.\begin{array}{c} \mathrm{c}+\mathrm{b}=\mathrm{d}, \quad \dfrac{\mathrm{b}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{b}'}, \quad \dfrac{\mathrm{c}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{c}'}, \quad \dfrac{\mathrm{d}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{d}'}, \\ \dfrac{\mathrm{b}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{b}'}, \quad \dfrac{\mathrm{c}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{c}'}, \quad \dfrac{\mathrm{d}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{d}'}, \end{array}\right\} \qquad (79),$$

we shall have the first member of the formula (77) equal to $\frac{\mathrm{a}'}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{d}'} = \frac{\mathrm{a}'}{\mathrm{d}'}$ = the second member of that formula; it will therefore be equal to $\frac{\mathrm{d}''}{\mathrm{a}'}$, and consequently will be shown to be $= \frac{\mathrm{c}''}{\mathrm{a}'} + \frac{\mathrm{d}''}{\mathrm{a}'} = \frac{\mathrm{a}'}{\mathrm{c}'} + \frac{\mathrm{a}'}{\mathrm{b}'}$ = the third member of that formula, if we can show that the conditions (79) give the relation

$$\mathrm{d}'' = \mathrm{c}'' + \mathrm{b}'' \qquad (80).$$

Now those conditions show that the line a is common to the planes of b, b′, and c, c′, and that it bisects the angle between b and b′, and also the angle between c and c′; therefore the mutual inclination of the lines b′ and c′ is equal to the mutual inclination of b and c; while the lengths of the two former lines are, by the same conditions, inversely proportional to those of the two latter. An on pursuing this geometrical reasoning, in combination with the definitional meanings of the symbolic equations (79), it appears easily that the mutual inclinations of the lines b″, c″, d″, are equal to those of b′, c′, d′, and therefore to those of b, c, d; while the lengths of b″, c″, d″ are inversely proportional to those of b′, c′, d′, therefore directly proportional to the lengths of b, c, d: since then the line d is the symbolic sum of b and c, or the diagonal of a parallelogram described with those two lines as adjacent sides, it follows that the line d″ is similarly related to b″ and c″, or that the relation (80) holds good. The formula (77) is therefore shown to be true: and although we have not *yet* proved that the multiplication of two geometrical fractions is *always* a *distributive* operation, we see at least that either factor may be distributed into two partial factors, and that the sum of the two partial products will give the total product, whenever either total factor and the two parts of the other factor are *co-linear*; that is, whenever the planes of these three fractions are *parallel to any common line*, such as the line a in the formulæ (76) (77): the *plane* of a geometrical fraction being one which contains or is parallel to the numerator and denominator thereof. A *scalar* fraction, being the quotient of two parallel lines, of which either may be transported without altering its direction to any other position in space while both may revolve together, may be regarded as having an entirely *indeterminate plane*, which may thus be rendered parallel to any arbitrary line; we shall therefore always satisfy the condition of *co-linearity*, by distributing either or both of two factors into their scalar and vector parts, and may consequently write,

$$\begin{aligned}\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}} &= \left(\mathrm{V}\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) \\ &= \left(\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\frac{\mathrm{f}}{\mathrm{e}}\right) \\ &= \left(\mathrm{V}\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{V}\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\frac{\mathrm{f}}{\mathrm{e}}\right) \qquad (81);\end{aligned}$$

or more concisely,

$$(\beta + b)(\alpha + a) = \beta\alpha + \beta a + b\alpha + ba \qquad (82),$$

if we denote, as in a former article, vectors by greek and scalars by italic letters, and omit the mark of multiplication between any two successive letters of these two kinds, or between sums of such letters, when those sums are enclosed in parentheses. But the multiplication of scalars is effected, as we have seen, by the ordinary rules of algebra; and to multiply a vector by a scalar, or a scalar by a vector, is easily shown, by the definitions already laid down, to be equivalent to multiplying by the scalar, on the plan of the sixth article, either the index or the numerator of the vector, without altering the denominator of that vector: thus, in the second member of (82), the term ba is a known scalar, and the terms $b\alpha$, βa are known vectors, if the partial factors a, b, α, β be known: in order therefore to apply the equation (82), which in its form agrees with ordinary algebra, to any question of multiplication of any two geometrical fractions, it is sufficient to know how to interpret generally the remaining term $\beta\alpha$, or the product of one vector by another. For this purpose we may always conceive the index $\mathrm{I}\beta$ of the vector β to be the sum of two other indices, which shall be respectively parallel and perpendicular to the index $\mathrm{I}\alpha$ of the other vector α, as follows:

$$\mathrm{I}\beta' \parallel \mathrm{I}\alpha, \quad \mathrm{I}\beta'' \perp \mathrm{I}\alpha, \quad \mathrm{I}\beta'' + \mathrm{I}\beta' = \mathrm{I}\beta \qquad (83);$$

and then the vector β itself will be, by the last article, the sum of the two new vectors β' and β'', and the planes of these two new vector fractions will be respectively parallel and perpendicular to the plane of the vector fraction α; consequently, the three fractions β', β'', α will be co-linear, and we shall have, by the principle (76),

$$\beta\alpha = (\beta' + \beta'')\alpha = \beta'\alpha + \beta''\alpha \qquad (84).$$

The problem of the multiplication of *any two* vectors is thus decomposed into the two simpler problems, of multiplying first *two parallel*, and secondly *two rectangular, vectors* together. If then we merely wish to separate the scalar and the vector parts, it is sufficient to observe that if, in the general formula (47), for the multiplication of any two fractions, we suppose the factors to be parallel vectors, then the line a is perpendicular to both b and c, and is also co-planar with them, so that they are necessarily parallel to each other, and the product $\dfrac{\text{c}}{\text{b}}$ is a scalar; but if, in the same general formula, we suppose the factors to be rectangular vectors, then the three lines a, b, c are themselves mutually rectangular, and the product of the fractions is a vector. Thus, in the formula (84), the partial product $\beta'\alpha$ is a scalar, but the other partial product $\beta''\alpha$ is a vector; and we may write

$$\mathrm{S} \,.\, \beta\alpha = \beta'\alpha; \quad \mathrm{V} \,.\, \beta\alpha = \beta''\alpha \tag{85}$$

We may therefore, more generally, under the conditions (83), decompose the formula of multiplication (82) into the two following equations:

$$\left.\begin{aligned} \mathrm{S} \,.\, (\beta + b)(\alpha + a) &= \beta'\alpha + ba; \\ \mathrm{V} \,.\, (\beta + b)(\alpha + a) &= \beta''\alpha + \beta a + b\alpha \end{aligned}\right\} \tag{86}$$

Or we may write, for abridgment,

$$c = \beta'\alpha + ba; \quad \gamma = \beta''\alpha + \beta a + b\alpha \tag{87}$$

and then we shall have this other equation of multiplication,

$$\gamma + c = (\beta + b)(\alpha + a) \tag{88}$$

And thus the general *separation of the scalar and vector parts* of the product of any two geometrical fractions may be effected. But it seems proper to examine more closely into the separate meanings of the two partial products of vectors, denoted here by the two terms $\beta'\alpha$ and $\beta''\alpha$; which will be done in the two following articles.

Products of two Parallel Vectors; Geometrical Representations of the Square Roots of Negative Scalars.

12. It was shown, in the last article, that the product of any two parallel vectors, such as α and β', that is, the product of any two vectors of which the planes or the indices are parallel, is equal to a scalar. By pursuing the reasoning of that article, it is easy to show, farther, that this *scalar product of two parallel vectors* is equal to the *product of the numbers* which express the lengths of the two parallel indices; this numerical product being taken with a *negative* or with a *positive* sign, according as these indices are *similar* or *opposite* in direction. In fact, in the general formula $\dfrac{\text{c}}{\text{a}} \times \dfrac{\text{a}}{\text{b}} = \dfrac{\text{c}}{\text{b}}$, we have now $\text{b} \perp \text{a}$, $\text{c} \parallel \text{b}$; the length of c is to the length of b, in a ratio compounded of the length of c to that of a, and of the ratio of the length of a to that of b; and the direction of c is opposite or similar to that of b, according as the two quadrantal rotations in one common plane, from b to a, and from a to c, are performed right-handedly round the same index, or round opposite indices.

We know then perfectly how to interpret the product of any two parallel vectors; and, as a case of such interpretation, if we agree to say that the product of any two equal fractions is the *square* of either, and to write

$$\frac{\text{b}}{\text{a}} \times \frac{\text{b}}{\text{a}} = \left(\frac{\text{b}}{\text{a}}\right)^2 \tag{89}$$

whatever two lines may be denoted by a and b, we see that, in the present system, the *square of a vector is always a negative scalar*, namely the negative of the square of the number which denotes the length of the index of the vector; in such a manner that, for any vector α, we shall have the equation

$$\alpha^2 = -\overline{\alpha}^2 \qquad (90),$$

if we agree to denote by the symbol $\overline{\alpha}$ that positive or absolute number which expresses the *length of the index* Iα. We have then, reciprocally,

$$\overline{\alpha}^2 = -\alpha^2 \qquad (91);$$

and may therefore write

$$\overline{\alpha} = \sqrt{(-\alpha^2)} \qquad (92),$$

$-\alpha^2$being here a positive number (because α^2 is negative), and $\sqrt{(-\alpha^2)}$ being its positive or absolute *square root*, which is an entirely *determined* (and real) *number*, when the vector α, or even when the length of its index, is determined. But although we might be led to write, in like manner, from (90), the equation

$$\alpha = \sqrt{(-\overline{\alpha}^2)} \qquad (93),$$

yet the same principles prove that this expression, which may denote generally any *square root of a negative number*, by a suitable choice of the positive number $\overline{\alpha}$, is equal to a *vector* α, of which the index Iα has indeed a *determined length*, but has an entirely *undetermined direction*; the symbol in the second member of the equation (93) may therefore receive (in the present system) infinitely many different geometrical representations, or constructions, though they have all one common character; and it will be a little more consistent with the analogies of ordinary algebra to write the equation under the form

$$\alpha = (-\overline{\alpha}^2)^{\frac{1}{2}} \qquad (94),$$

using a fractional exponent which suggests a certain degree of indeterminateness, rather than a radical sign which it is often convenient to restrict to one determined value. Thus, for example, the symbol $(-1)^{\frac{1}{2}}$ or the *square root of negative unity*, will, in the present system, denote, or be geometrically constructed by, *any vector of which the index is equal to the unit of length*; that is, any geometrical fraction of which the numerator and the denominator are lines equal to each other in length, but perpendicular to each other in direction. And we see that the geometrical principle, on which this conclusion ultimately depends, is simply this: that *two successive and similar quadrantal rotations, in any arbitrary plane, reverse the direction* of any straight line in that plane. Mr. Warren, confining himself to the consideration of lines in *one fixed plane*, has been led to attribute to his geometrical representations of the square roots of negative numbers, *one fixed direction*, or rather axis, perpendicular to that other axis on which he represents square roots of positive numbers. And other authors, both before and since the publication of Mr. Warren's work,* seem to have been in like manner

* *Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*, by the Rev. John Warren, Cambridge, 1828. See also Dr. Peacock's Treatises on Algebra, and his Report to the British Association, containing references to other works.

disposed to represent positive or negative numbers by lines in some one direction, or in the direction opposite, but symbols of the form $a\sqrt{(-1)}$ by lines perpendicular thereto. Such is at least the impression on the mind of the present writer, produced perhaps by an insufficient acquaintance with the works of those who have already written on this class of subjects. It will however be attempted to show, in a future article of this paper, that the geometrical fractions which have been called *vectors*, in the present and in former articles, may be symbolically equated to their own indices; and that thus *every straight line having direction in space* may properly be looked upon *in the present system* as *a geometrical representation of a square root of a negative number*; while positive and negative numbers are in the same system regarded indeed as belonging to one common *scale* of progression, from $-\infty$ to $+\infty$, but to a scale which is not to be considered as having any one direction rather than any other, in tridimensional space.

Products of two Rectangular Vectors; Non-commutativeness of the Factors, in the general Multiplication of two Geometrical Fractions.

13. The reasoning by which it was shown, in the 11th article, that the *product* $\beta''\alpha$ *of any two rectangular vectors*, α and β'', is *itself a vector*, may be continued so as to show that the number expressing the length of the index of this vector product is the product of the numbers which express the lengths of the indices of the factors; or that, in a notation similar to one employed in the last article,

$$\overline{\beta''\alpha} = \overline{\beta''}\,\overline{\alpha}, \quad \text{when, } \mathrm{I}\beta'' \perp \mathrm{I}\alpha \qquad (95);$$

and therefore that, by the principle (92), for the same case of *rectangular vectors*, we have the formula

$$\sqrt{\{-(\beta''\alpha)^2\}} = \sqrt{(-\beta''^2)}\sqrt{(-\alpha^2)} \qquad (96).$$

Also in the general formula of multiplication $\frac{\mathrm{c}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{b}}$, the three lines a, b, c compose here a rectangular system; and therefore the *index of the product* is parallel to the line a, and is consequently *perpendicular to the indices of the two factors*; $\mathrm{I}\,.\,\beta''\alpha$ is therefore perpendicular to both $\mathrm{I}\beta''$ and $\mathrm{I}\alpha$; a conclusion which may be extended by (83) and (85) to the multiplication of *any two vectors*, so that we may write generally,

$$\mathrm{I}\,.\,\beta\alpha \perp \mathrm{I}\beta; \quad \mathrm{I}\,.\,\beta\alpha \perp \mathrm{I}\alpha \qquad (97).$$

Again, we are allowed to suppose, in applying the same general formula of multiplication to the same case of rectangular vectors, that the index $\mathrm{I}\alpha$ of the multiplicand $\frac{\mathrm{a}}{\mathrm{b}}$ is not only parallel to the line c, but similar (and not opposite) in direction to that line; in such a manner that the rotation round c from b to a is positive: and then the rotation round b from a to c is positive, and so is the rotation round a from c to b, and also that round $-$a from b to c; therefore the index $\mathrm{I}\beta''$ of the multiplier is similar in direction to +b, and the index $\mathrm{I}\,.\,\beta''\alpha$ of the product is similar in direction to $-$a; consequently *the rotation round the index of the product, from the index of the multiplier to that of the multiplicand, is positive.* And although this last result has only been proved here for the case of two rectangular vectors, yet it may easily be shown, by the principles of the 11th article, to extend to the multiplication of two general fractions. For, in the notation of that article, γ denoting the vector part of the product of any two such fractions, we have, by (87)

$$\mathrm{I}\gamma = \mathrm{I} \,.\, \beta''\alpha + a\mathrm{I}\beta + b\mathrm{I}\alpha \qquad (98);$$

$\mathrm{I}\gamma$ is therefore the symbolic sum of $\mathrm{I} \,.\, \beta''\alpha$ and of two other lines which are respectively parallel to the indices of the vector parts of the two factors, and which consequently have their sum co-planar with those indices, and therefore also co-planar, by (83), with $\mathrm{I}\beta''$ and $\mathrm{I}\alpha$; consequently $\mathrm{I}\gamma$ and $\mathrm{I} \,.\, \beta''\alpha$ both lie at the same side of the plane of $\mathrm{I}\alpha$ and $\mathrm{I}\beta''$; and therefore the rotation round $\mathrm{I}\gamma$, like that around $\mathrm{I} \,.\, \beta''\alpha$, from $\mathrm{I}\beta''$ to $\mathrm{I}\alpha$, and consequently from $\mathrm{I}\beta$ to $\mathrm{I}\alpha$, is positive. Hence also the rotation round $\mathrm{I}\beta$ from $\mathrm{I}\alpha$ to $\mathrm{I}\gamma$ is positive; that is to say, in the multiplication of two general geometrical fractions, *the rotation round the index of the vector part of the multiplier, from that of the multiplicand to that of the product, is positive*; from which may immediately be deduced a remarkable consequence, already alluded to by anticipation in the 8th article, namely—that the *multiplication of two general geometrical fractions is not a commutative operation*, or that the *order of the factors is not in general indifferent*; since the index of the vector part of the product lies at one or at another side of the plane of the indices of the vector parts of the two factors, according as those factors are taken in one or in the other order. We have, for example, by the present article, a relation of *opposition* of signs between the products of two *rectangular* vectors, taken in two opposite orders; which relation may be expressed by the following *equation of perpendicularity*,

$$\alpha\beta'' = -\beta''\alpha, \quad \text{when, } \mathrm{I}\beta'' \perp \mathrm{I}\alpha \qquad (99).$$

But in the case where the indices of the vector parts α and β of two fractional factors are *parallel* which includes the case where either of those indices vanishes, the corresponding factor becoming then a scalar), the part β'' of the vector β vanishes, and the latter vector reduces itself by (83) to its other part β'; so that in *this* case, by the results of the last article, the orders of the factors is indifferent, and the operation of multiplication is commutative: and thus we may write, as the *equation of parallelism* between two vectors,

$$\alpha\beta' = \beta'\alpha, \quad \text{when, } \mathrm{I}\beta' \parallel \mathrm{I}\alpha \qquad (100).$$

It is easy to infer hence, by (84) and (77), that in the more general case of the multiplication of any two vectors α and β, we may write, instead of (85), the following formulæ for the separation of the scalar and vector parts of the product:

$$\left.\begin{aligned} \mathrm{S} \,.\, \beta\alpha &= \tfrac{1}{2}(\beta\alpha + \alpha\beta) = \mathrm{S} \,.\, \alpha\beta \\ \mathrm{V} \,.\, \beta\alpha &= \tfrac{1}{2}(\beta\alpha - \alpha\beta) = -\mathrm{V} \,.\, \alpha\beta \end{aligned}\right\} \qquad (101),$$

with corresponding formulæ instead of (86), which give

$$(\beta + b)(\alpha + a) - (\alpha + a)(\beta + b) = \beta\alpha - \alpha\beta \qquad (102),$$

the second member of this last equation being a vector different from 0, unless it happen that the planes (or the indices) of the vectors α and β are parallel to each other. Finally, we may here observe that in virtue of the principles and definitions already laid down, *the length of the index* ($\mathrm{I} \, . \, \beta\alpha$) *of the vector part of the product of any two vectors bears to the unit of length the same ratio which the area of the parallelogram under the indices* ($\mathrm{I}\beta$ and $\mathrm{I}\alpha$) *of the factors bears to the unit of area* the *direction* of this index of the product being also (as we have seen) *perpendicular to the plane* of the indices of the factors, and therefore to the plane of the parallelogram under them; and being changed to its own *opposite* when the order of the factors is inverted, which *inversion* of their order may be considered as corresponding to a *reversal of the face* of the parallelogram: for all which reasons, there appears to be a propriety in considering this index of the vector part of a product of any two vectors as a symbolical representation of this parallelogram under the indices of the factors, and in writing the symbolical equation

$$\mathrm{I} \, . \, \beta\alpha = \square \, (\mathrm{I}\beta, \mathrm{I}\alpha) \tag{103}$$

It will be remembered that the indices $\mathrm{I}(\beta+\alpha)$, $\mathrm{I}(\beta-\alpha)$, of the sum and difference of the same two vectors, are symbolically equal to two different diagonals of the same parallelogram, by former articles of this paper.

On the Distributive Character of the Operation of Multiplication, as performed generally on Geometrical Fractions.

14. We are now prepared to extend the formulæ (76), (77), respecting the multiplication of sums of geometrical fractions; and to shew that similar results hold good, even when the conditions of colinearity, assumed in those two formulæ, is no longer supposed to be satisfied. That is, the two equations

$$\left(\frac{\mathrm{h}}{\mathrm{g}} + \frac{\mathrm{f}}{\mathrm{e}}\right) \times \frac{\mathrm{k}}{\mathrm{i}} = \left(\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{k}}{\mathrm{i}}\right) + \left(\frac{\mathrm{f}}{\mathrm{e}} \times \frac{\mathrm{k}}{\mathrm{i}}\right) \tag{104}$$

$$\frac{\mathrm{k}}{\mathrm{i}} \times \left(\frac{\mathrm{h}}{\mathrm{g}} + \frac{\mathrm{f}}{\mathrm{e}}\right) = \left(\frac{\mathrm{k}}{\mathrm{i}} \times \frac{\mathrm{h}}{\mathrm{g}}\right) + \left(\frac{\mathrm{k}}{\mathrm{i}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) \tag{105}$$

can both be shown to be true, whatever may be the lengths and directions of the six lines e, f, g, h, i, k; although, by the general non-commutativeness of geometrical fractions as factors, which was pointed out in the last article, the expressions contained in these two equations are not to be confounded with each other.

Making for this purpose

$$\left.\begin{array}{c} \frac{\mathrm{f}}{\mathrm{e}} = \beta_1 + b_1, \quad \frac{\mathrm{h}}{\mathrm{g}} = \beta_2 + b_2, \quad \frac{\mathrm{k}}{\mathrm{i}} = \alpha + a, \\ \mathrm{I}\beta_1' \parallel \mathrm{I}\alpha, \quad \mathrm{I}\beta_1'' \perp \mathrm{I}\alpha, \quad \mathrm{I}\beta_1'' + \mathrm{I}\beta_1' = \mathrm{I}\beta_1, \\ \mathrm{I}\beta_2' \parallel \mathrm{I}\alpha, \quad \mathrm{I}\beta_2'' \perp \mathrm{I}\alpha, \quad \mathrm{I}\beta_2'' + \mathrm{I}\beta_2' = \mathrm{I}\beta_2, \\ \beta_2' + \beta_1' = \beta', \quad \beta_2'' + \beta_1'' = \beta'', \quad \beta_2 + \beta_1 = \beta, \quad b_2 + b_1 = b, \end{array}\right\} \tag{106}$$

the conditions (83) will be satisfied; and if we still assign to γ and c the meanings (87), the equation (88) will hold good, and $\gamma + c$ will be an expression for the first member of (104). Making also, in imitation of (87),

$$\left.\begin{array}{ll} c_1 = \beta_1'\alpha + b_1 a, & \gamma_1 = \beta_1''\alpha + \beta_1 a + b_1\alpha, \\ c_2 = \beta_2'\alpha + b_2 a, & \gamma_2 = \beta_2''\alpha + \beta_2 a + b_2\alpha, \end{array}\right\} \qquad (107),$$

the second member of the same equation (104) becomes, by the principles of the 11th article, $(\gamma_2 + c_2) + (\gamma_1 + c_1)$; and the equation resolves itself into the two following,

$$c = c_2 + c_1, \quad \gamma = \gamma_2 + \gamma_1 \qquad (108);$$

which are easily seen to reduce themselves to these two,

$$(\beta_2' + \beta_1')\alpha = \beta_2'\alpha + \beta_1'\alpha; \quad (\beta_2'' + \beta_1'')\alpha = \beta_2''\alpha + \beta_1''\alpha \qquad (109);$$

the one being an equation between scalars, and the other between vectors. In like manner the equation (105) may be shown to depend on the two following equations, less general than itself, but of the same form,

$$\alpha(\beta_2' + \beta_1') = \alpha\beta_2' + \alpha\beta_1'; \quad \alpha(\beta_2'' + \beta_1'') = \alpha\beta_2'' + \alpha\beta_1'' \qquad (110).$$

And since, by (101), the three scalar products in the equations (110) are respectively equal, and the three vector products are respectively opposite (in their signs) to the corresponding products in the equations (109), it is sufficient to prove either of these two pairs of equations; for example, the pair (110). Now the first equation of this pair is true, because the scalars denoted by the three products $\alpha\beta_1'$, $\alpha\beta_2'$, $\alpha(\beta_2'+\beta_1')$, are proportional, both in their magnitudes and in their signs, to the indices of the parallel vectors β_1', β_2', $\beta_2'+\beta_1'$; and the second equation of the same pair is true, because the indices of the vectors denoted by the three other products $\alpha\beta_1''$, $\alpha\beta_2''$, $\alpha(\beta_2''+\beta_1'')$ are formed from the indices of the three coplanar vectors β_1'', β_2'', $\beta_2''+\beta_1''$, by causing the three latter indices to revolve together, as one system, in their common plane, round the index $\mathrm{I}\alpha$, their lengths being at the same time changed (if at all) in one common ratio, namely, in that of $\overline{\alpha}$ to 1. The formulæ (104) (105) are therefore proved to be true; and the same reasoning shows, that in any multiplication of two geometrical fractions, either of the fractions may be *distributed* into *any number* of parts, and that the sum of the partial products will be equal to the total product: so that we may write, generally,

$$\left(\Sigma\frac{\mathrm{k}}{\mathrm{i}}\right) \times \left(\Sigma\frac{\mathrm{f}}{\mathrm{e}}\right) = \Sigma\left(\Sigma\frac{\mathrm{k}}{\mathrm{i}} \times \Sigma\frac{\mathrm{f}}{\mathrm{e}}\right) \qquad (111).$$

The *multiplication of geometrical fractions* is therefore a *distributive operation;* although it has been shown to be *not*, in general, a *commutative* one.

15. Proceeding now, with the help of the distributive property established in the last article, and of the principle that a product is multiplied by a scalar when any one of its factors is multiplied thereby, to prove that the multiplication of geometrical fractions is general an *associative* operation, or that the formula

$$\frac{\mathrm{k}}{\mathrm{i}} \times \left(\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) = \left(\frac{\mathrm{k}}{\mathrm{i}} \times \frac{\mathrm{h}}{\mathrm{g}}\right) \times \frac{\mathrm{f}}{\mathrm{e}} \tag{112}$$

holds good for *any three fractions* (with other formulæ of the same sort for more fractional factors than three), it will be sufficient to prove that the formula is true for *any three vectors*; or that we may write generally

$$\gamma \times \beta\alpha = \gamma\beta \times \alpha \tag{113}$$

the vector γ being not here obliged to satisfy the equation (87); we may even content ourselves with proving that the equation (113) is true in the two following cases, namely first, when any two of the three vectors are parallel; and secondly, when all three are rectangular to each other. The first case may be expressed by the three following equations as its types—

$$\beta \times \beta\alpha = \beta\beta \times \alpha \tag{114}$$

$$\beta \times \alpha\beta = \beta\alpha \times \beta \tag{115}$$

$$\alpha \times \beta\beta = \alpha\beta \times \beta \tag{116}$$

and the second case may be expressed by the equation

$$\alpha\beta \times \beta\alpha = (\alpha\beta \times \beta) \times \alpha, \quad \text{when } \beta \perp \alpha \tag{117}$$

because, under this last condition, $\alpha\beta$ is, by Art. 13, a vector, rectangular to both α and β. Under the same condition we may, by (99), change $\alpha\beta$ to $-\beta\alpha$; therefore the first member of the equation (117) may be equated to $-(\beta\alpha)^2$, and consequently, by (96), to $(-\beta^2) \times (-\alpha^2) = \beta^2 \times \alpha^2 = \beta^2\alpha \times \alpha = (\alpha \times \beta\beta) \times \alpha$, because β^2 or $\beta\beta$ is, by Art. 12, a scalar; thus we may make (117) depend on (116), which again depends on (114), and on the following equation,

$$\beta \times \beta\alpha = \alpha\beta \times \beta \tag{118}$$

Equations (118) and (115) may both be proved by observing that, by Art. 13, whatever two vectors may be denoted by α and β, we have the expressions

$$\left.\begin{aligned} \beta\alpha &= \mathrm{S}\,.\,\beta\alpha + \mathrm{V}\,.\,\beta\alpha, \\ \alpha\beta &= \mathrm{S}\,.\,\beta\alpha - \mathrm{V}\,.\,\beta\alpha, \end{aligned}\right\} \tag{119}$$

with the relations

$$\left.\begin{aligned} \beta \times \mathrm{S}\,.\,\beta\alpha - \mathrm{S}\,.\,\beta\alpha \times \beta = 0, \\ \beta \times \mathrm{V}\,.\,\beta\alpha + \mathrm{V}\,.\,\beta\alpha \times \beta = 0, \end{aligned}\right\} \tag{120}$$

It remains then to prove the equation (114); and it is sufficient to prove this for the case when α and β are two rectangular vectors. But, in this case, $\beta\alpha$ is a vector formed from α by causing its index to revolve right-handedly through a right angle round the index $\mathrm{I}\beta$, to which it is perpendicular, changing at the same time in general the length of this revolving index from $\overline{\alpha}$ to $\overline{\beta} \times \overline{\alpha}$; and the repetition of this process, directed by the symbol $\beta \times \beta\alpha$, conducts to a new vector, of which the index is in direction opposite to the original direction of $\mathrm{I}\alpha$, and in length equal to $\overline{\beta}^2 \times \overline{\alpha}$: this new vector may therefore be otherwise denoted by $-\overline{\beta}^2 \times \alpha$, or by $\beta^2 \times \alpha$, and the eqaution (114) is true. The equations (113) and (112) are therefore also true; and since the latter formula may easily be extended to any number of fractional factors, we are now entitled to conclude what it was in the beginning of the present article proposed to prove; namely, that *the multiplication of geometrical fractions is always an associative operation*: as the addition of fractions, and the addition of lines, have in former articles been shown to be. In other words, any number of successive fractional factors may be *associated* or grouped together by multiplication (without altering their order) into a single product, and this product substituted as a single factor in their stead; a result which constitutes a new agreement (the more valuable on account of the absence of identity in some other important respects), between the *rules of operation* of ordinary algebra, and those of the present Symbolical Geometry.

Other forms of the Associative Principle of Multiplication.

16. By the principles already established respecting the transformation of geometrical fractions, any three such fractions, $\frac{\mathrm{f}}{\mathrm{e}}$, $\frac{\mathrm{h}}{\mathrm{g}}$, $\frac{\mathrm{k}}{\mathrm{i}}$, may be so prepared that the numerator of the first shall be in the plane of the second, and that the numerator of the second shall coincide with the denominator of the third; we may, therefore, without diminishing the generality of the theorem expressed by the formula (112), suppose that the line i is equal to h, and that the fourth proportional to g, h, f, is a new line l; and with this preparation the associative principle of multiplication, established in the foregoing article, may be put under the following form, in which the mark of multiplication between two fractional factors is omitted for the sake of conciseness:

$$\text{if } \frac{\mathrm{h}}{\mathrm{g}} = \frac{\mathrm{l}}{\mathrm{f}}, \quad \text{then } \frac{\mathrm{k}}{\mathrm{h}}\frac{\mathrm{l}}{\mathrm{e}} = \frac{\mathrm{k}}{\mathrm{g}}\frac{\mathrm{f}}{\mathrm{e}} \qquad (121);$$

that is to say, *the product of any two geometrical fractions will remain unaltered* in value, or will still continue to represent the same third fraction, *if the denominator of the multiplier and the numerator of the multiplicand be changed to any new lines to which they are proportional*, or with which they form a *symbolic analogy*, including a relation between *directions* as well as a proportion of lengths, of the kind considered in Mr. Warren's work, (and earlier by Argand and Français,) and in the seventh article of this paper. Reciprocally, by the associative principle, the former of the two equations (121) is in general a consequence of the latter, that is, if the product of two geometrical fractions be equal to the product of two other fractions of the same sort, and if the multipliers have a common numerator, and the multiplicands a common denominator, then the numerators of the two multiplicands and the denominators of the two multipliers are the antecedents and consequents of a symbolical proportion or analogy, of the kind considered in the seventh article: for we may write

$$\frac{\mathrm{h}}{\mathrm{g}} = \frac{\mathrm{h}}{\mathrm{k}} \left(\frac{\mathrm{k}}{\mathrm{g}} \frac{\mathrm{f}}{\mathrm{e}} \right) \frac{\mathrm{e}}{\mathrm{f}}, \quad \frac{\mathrm{h}}{\mathrm{k}} \left(\frac{\mathrm{k}}{\mathrm{h}} \frac{\mathrm{l}}{\mathrm{e}} \right) \frac{\mathrm{e}}{\mathrm{f}} = \frac{\mathrm{l}}{\mathrm{f}};$$

so that the first equation (121) may be obtained from the second, by suitably grouping or associating the factors.

Again, the same associative principle shows that

$$\text{if } \frac{\mathrm{c}}{\mathrm{c}'} = \frac{\mathrm{b}'}{\mathrm{b}} \frac{\mathrm{a}'}{\mathrm{a}}, \quad \text{then } \frac{\mathrm{c}}{\mathrm{b}'} = \frac{\mathrm{c}'}{\mathrm{a}} \frac{\mathrm{a}'}{\mathrm{b}}, \qquad (122);$$

for the first equation (122) may be replaced by the system of the three following equations,

$$\frac{\mathrm{a}'}{\mathrm{a}} = \frac{\mathrm{b}''}{\mathrm{a}''}, \quad \frac{\mathrm{b}'}{\mathrm{b}} = \frac{\mathrm{c}''}{\mathrm{b}''}, \quad \frac{\mathrm{c}'}{\mathrm{c}} = \frac{\mathrm{a}''}{\mathrm{c}''} \qquad (123);$$

of which the two last give, for the first member of the second equation (122), the expression

$$\frac{\mathrm{c}}{\mathrm{b}'} = \frac{\mathrm{c}'}{\mathrm{a}''} \frac{\mathrm{b}''}{\mathrm{b}},$$

which is equal to the second member of the same second equation (122), by the first of the three equations (123), and by the theorem (121): whenever, therefore, we meet an equation between one geometrical fraction and the product of two others, we are at liberty to *interchange the denominator of the product and the numerator of the multiplier*, provided that we at the same time *interchange the denominators of the two factors*; no change being made in the numerators of the product and the multiplicand. Conversely, this assertion respecting the liberty to make these interchanges, and the formula (122), to which the assertion corresponds, are modes of expressing the associative principle of multiplication; for by introducing the equations (123) we find that the theorem (122) conducts to the following relation, or *identity between the two ternary products of three fractions*, associated in two different ways, but with one common order of arrangement,

$$\frac{\mathrm{c}'}{\mathrm{a}''} \left(\frac{\mathrm{a}''}{\mathrm{a}} \frac{\mathrm{a}'}{\mathrm{b}} \right) = \left(\frac{\mathrm{c}'}{\mathrm{a}''} \frac{\mathrm{a}''}{\mathrm{a}} \right) \frac{\mathrm{a}'}{\mathrm{b}} \qquad (124);$$

in which last form, as in (112), the three factors multiplied together may represent any three geometrical fractions. We may also present the same principle under the form of the following theorem—

$$\text{if } \frac{\mathrm{c}'}{\mathrm{c}} \frac{\mathrm{b}'}{\mathrm{b}} \frac{\mathrm{a}'}{\mathrm{a}} = 1, \quad \text{then } \frac{\mathrm{c}'}{\mathrm{a}} \frac{\mathrm{a}'}{\mathrm{b}} \frac{\mathrm{b}'}{\mathrm{c}} = 1 \qquad (125);$$

and may derive from it, with the help of (123), the following value of a certain product of six fractional factors,

$$\frac{\mathrm{a}''}{\mathrm{c}''} \frac{\mathrm{c}}{\mathrm{a}} \frac{\mathrm{b}''}{\mathrm{a}''} \frac{\mathrm{a}}{\mathrm{b}} \frac{\mathrm{c}''}{\mathrm{b}''} \frac{\mathrm{b}}{\mathrm{c}} = 1 \qquad (126):$$

which must hold good whenever the three lines a, b, c are respectively coplanar with the three pairs $a''b''$, $b''c''$, $c''a''$. Finally, it may be stated here, as a theorem essentially equivalent to the associative principle of multiplication, although not expressly involving the product of two or more fractions, that *in the system of the six equations* of which those marked (123) are three, and of which the others are the three following analogous equations,

$$\frac{a}{c'} = \frac{a'''}{c'''}, \quad \frac{b}{a'} = \frac{b'''}{a'''}, \quad \frac{c}{b'} = \frac{c'''}{b'''} \qquad (127);$$

any five equations of the system include the sixth.

Geometrical Interpretation of the Associative Principle: Symbolic Equations between Arcs upon a Sphere: Theorem of the two Spherical Hexagons.

17. If we attended only to the *lengths* of the various lines compared, the associative principle of multiplication, under all the foregoing forms, would be nothing more than an easy and known consequence of a few elementary theorems respecting compositions of ratios of magnitudes. On the other hand it is permitted, in the present symbolical geometry, to assume at pleasure the *situations* of straight lines denoted by small roman letters, provided that the lengths and directions are preserved. The general theorem or property of multiplication, which has been expressed in various ways in the two foregoing articles, may therefore be regarded as being essentially a *relation, or system of relations, between the directions of certain lines in space.*

In this view of the subject no essential loss of generality (or at least none which cannot easily be supplied by known and elementary principles) will be sustained by supposing all the straight lines abc, $a'b'c'$, $a''b''c''$, $a'''b'''c'''$, efghikl, of the two last articles to be *radii of one sphere*, setting out from one *common origin* or centre O, and terminating in points upon one *common spheric surface*, which may be denoted respectively by the symbols ABC, $A'B'C'$, $A''B''C''$, $A'''B'''C'''$, EFGHIKL. In order more conveniently to study and express relations between points so situated, we may agree to say that two *arcs upon one sphere*, such as those from G to H and from F to L, are *symbolically equal*, when they are *equally long and similarly directed portions of the circumference of one great circle*; and may denote this *symbolical equality between arcs*, so called for the sake of suggesting that (like the symbolical equality between straight lines considered in the second article) it involves a relation of *identity of directions*, as well as a relation of equality of lengths, by writing any one of the three formulæ,

$$\left.\begin{array}{l} \frown \mathrm{LF} = \frown \mathrm{HG}, \\ \frown \mathrm{FL} = \frown \mathrm{GH}, \\ \frown \mathrm{LH} = \frown \mathrm{FG}, \end{array}\right\} \qquad (128);$$

of which the second may be called the *inverse*, and the third the *alternate* of the first. Any one of these three formulæ (128) will thus express the *same relation between the directions of the four coplanar radii*, namely the four lines f g h l, as that expressed by the first equation (121), or by its inverse, or its alternate equation; that is, by any one of the three following equations between geometrical fractions,

$$\frac{l}{f} = \frac{h}{g}, \quad \frac{f}{l} = \frac{g}{h}, \quad \frac{l}{h} = \frac{f}{g} \qquad (129).$$

The formulæ (128) express also the same relation between the same four directions, as that which would be expressed in a notation of a former article, by any one of the three following *symbolic analogies* between the same four lines,

$$\mathrm{l : f :: h : g}, \quad \mathrm{f : l :: g : h}, \quad \mathrm{l : h :: f : g} \qquad (130);$$

although it must not be forgotten that any one of the six latter formulæ, (129) and (130), expresses at the same time a proportion between the lengths of four straight lines, not generally equal to each other, which is not expressed by any one of the three former symbolical equations (128), between pairs of arcs upon a sphere. In this notation (128), the last form of the associative principle of multiplication which was assigned in the foregoing article, so far as it relates to directions only, may be expressed by saying that *any one of the six following symbolical equations between arcs is a consequence of the other five*,

$$\left.\begin{array}{l} \frown \mathrm{A'A} = \frown \mathrm{B''A''}, \\ \frown \mathrm{B'B} = \frown \mathrm{C''B''}, \\ \frown \mathrm{C'C} = \frown \mathrm{A''C''}, \end{array}\right\} \qquad (131);$$

$$\left.\begin{array}{l} \frown \mathrm{BA'} = \frown \mathrm{B'''A'''}, \\ \frown \mathrm{CB'} = \frown \mathrm{C'''B'''}, \\ \frown \mathrm{AC'} = \frown \mathrm{A'''C'''}, \end{array}\right\} \qquad (132).$$

Regarding *any six points* upon a spheric surface, in *any one order* of succession, as the *six corners of a spherical hexagon* (which may have re-entrant angles, and of which two or more sides may cross each other without being prolonged), we may speak of the arcs joining *successive corners* as the *sides*; those joining *alternate* corners, as the *diagonals*; and those joining *opposite* corners, as the *diameters* of this hexagon: the first side, first diagonal, and first diameter, respectively, being those three arcs which are drawn from the first corner to the second, third and fourth corners of the figure. With this phraseology, the form just now obtained for the result of the two foregoing articles may be expressed as a relation between two spherical hexagons, $\mathrm{AA'BB'CC'}$, $\mathrm{A''A'''B''B'''C''C'''}$, and may be enunciated in words as follows: *If five successive sides of one spherical hexagon be respectively and symbolically equal to five successive diagonals of another spherical hexagon, the sixth side of the first hexagon will be symbolically equal to the sixth diagonal of the second hexagon.* This theorem of spherical geometry, which may be called, for the sake of reference, the *theorem of the two hexagons*, is therefore a consequence, and may be regarded as an interpretation of the associative principle of multiplication: and conversely, in all applications to spherical geometry, and generally in all investigations respecting relations between the directions of straight lines in space, the associative principle of multiplication may be replaced by the theorem of the two spherical hexagons.

Other Interpretation of the Associative Principle of Multiplication: Theorem of the two Conjugate Transversals of a Spherical Quadrilateral (which are the Cyclic Arcs of a circumscribed Spherical Conic).

18. The theorem of the two hexagons gives also the following theorem: If upon each of the four sides of a spherical quadrilateral, or on that side prolonged, a portion be taken

commedial with the side (two arcs being said to be *commedial* when the have one common point of bisection); and if four extreme points of the four portions thus obtained be ranged on one transversal arc of a great circle, in such a manner that the part of this arc comprised between the first and third sides is commedial with the part comprised between the second and fourth: then the four other extremities of the same four portions will be ranged on another great circle; and the parts of this second or *conjugate* transversal, which are intercepted respectively by the same two pairs of opposite sides of the quadrilateral, will be in like manner commedial with each other.

For let the corners of the quadrilateral be denoted by the letters A, B, C, D, and let the side from A to B be cut in two points A' and B'', while the three other sides are cut in three other pairs of points, which may be called B' and C'', C' and D'', and D' and A'' respectively. Then, if the arcs from A' to C' and from B' to D' be commedial portions of one common great circle, or of a first transversal arc, the arcs from A' to B' and from D' to C' will be *symbolically equal arcs*, in the sense of the preceding article; and therefore, in the notation of that article, we may now write the equation

$$\frown B'A' = \frown C'D' \qquad (133).$$

In like manner the conditions, that the four portions of the sides of the quadrilateral shall be commedial with the sides themselves, give the four other equations of the same kind,

$$\left.\begin{array}{ll} \frown A'A = \frown BB''; & \frown B'B = \frown CC''; \\ \frown C'C = \frown DD''; & \frown D'D = \frown AA''. \end{array}\right\} \qquad (134).$$

Hence, by alternation and inversion, we find that the five successive sides

$$\frown AB'', \quad \frown D'A, \quad \frown C'D', \quad \frown CC', \quad \frown C''C,$$

of the spherical hexagon $B''AD'C'CC''$ are respectively and symbolically equal to the five successive diagonals

$$\frown A'B, \quad \frown DA'', \quad \frown B'A', \quad \frown D''D, \quad \frown BB',$$

of the other hexagon $BA''A'DB'D''$; and therefore, by the theorem of the two hexagons, the sixth side of the former figure must be symbolically equal to the sixth diagonal of the latter; that is, we may write the symbolical equation,

$$\frown B''C'' = \frown A''D'' \qquad (135).$$

But this expresses a relation equivalent to the following, that the two arcs from A'' to C'' and from B'' to D'' are commedial portions of one common great circle, or second transversal arc, which was the thing to be proved.

Reciprocally, the associative principle of geometrical multiplication, in so far as it relates to the directions of straight lines in space, may be expressed by the assertion that the symbolical equation between arcs (135) is a consequence of the five other equations of the same kind (133) and (134); this principle of symbolical geometry may therefore be so interpreted as to coincide with the foregoing *theorem of the two conjugate transversals* of a spherical quadrilateral, instead of the theorem of the two spherical hexagons. It is easy to see that to a given quadrilateral correspond infinitely many such pairs of conjugate transversal arcs; and those readers who are familiar with the theory of *spherical conics** will recognise in these conjugate transversals, A′B′C′D′, A″B″C″D″, the two *cyclic arcs* of such a conic, circumscribed about the proposed quadrilateral ABCD; but it suits better the plan of this communication on symbolical geometry to pass at present to another view of the subject.

It may however be noticed here, that in the first of the two hexagons already mentioned, *any two pairs of opposite sides intercept commedial portions of either of the two sides remaining*; and that the associative principle asserts that *if* a spherical hexagon have *five* of its sides thus *cut commedially*, the *sixth* side also will be cut in the same way. Or, because the two sets of alternate diagonals of the second hexagon are sides of two triangles, which have for their corners the alternate corners of this hexagon, we may in another way eliminate this second hexagon, and may express the same principle of spherical geometry by saying, that *if one set of alternate sides of a* (first) *spherical hexagon*, taken in their order, (as first, third, and fifth), *be respectively and symbolically equal to the three successive sides of a triangle*, then the *other set of alternate sides of the same hexagon will be* in like manner *symbolically equal to the sides of another triangle.* This last interpretation of the associative principle is even more immediately suggested than any other, by the forms of the equations (131) (132); in the notation of the present article, *the two triangles* are BA′B′ and A″DD″, which may be considered as having their *bases* A′B′ and A″D″ *on the two cyclic arcs* above alluded to, while their *vertical angles* at B and D may be said to be *angles in the same segment* (or in alternate segments) *of the spherical conic*: since, by (134), the two arcual sides BA′, BB′ of the one angle intersect respectively the two sides DA″, DD″ of the other angle, in the points A and C, which points of intersection, as well as the vertices B and D, are corners of the quadrilateral

* The plane of the first side of the quadrilateral, or the plane of OAB, if O denote the centre of the sphere, is cut by the plane of the first transversal arc in the radius A′O, and by the plane of the second transversal arc in the radius B″O. Thus the four plane faces of the tetrahedral angle, of which the four edges are the four radii from O to the four corners A, B, C, D of the quadrilateral, are cut by any secant plane parallel to the plane of the first transversal arc in four indefinite straight lines, which are respectively parallel to the four other radii A′O, B′O, C′O, D′O of the sphere; and consequently, in virtue of the equation (133), between the arcs which these last radii include, these four new lines in one common secant plane have the angular relation required for their being the (prolonged) sides of a (plane) quadrilateral inscribed in a circle; therefore the four edges of the same tetrahedral angle are cut by the same secant plane in points which are on the circumference of a circle; therefore they are edges or sides of a cone which has this circle for its base, and has its vertex at the centre of the sphere. But the intersection of such a cone with such a concentric sphere is called a *spherical conic*; a plane through its vertex, parallel to its circular base, is called a *cyclic plane*; and the intersection of this latter plane with the sphere has received the designation of a *cyclic arc.* Therefore the first transversal arc A′B′C′D′ is (as asserted in the text) a cyclic arc of a spherical conic circumscribed about the quadrilateral ABCD: and by a reasoning of exactly the same kind it may be proved, that the second transversal A″B″C″D″ is another cyclic arc of the same conic, or that its plane is a second cyclic plane, being parallel to the plane of another (or *subcontrary*) circular section.

inscribed in that spherical conic.

Symbolical Addition of Arcs upon a Sphere; Associative and Non-commutative Properties of such Addition.

19. The foregoing geometrical interpretations of the associative principle or property of the multiplication of geometrical fractions, may assist us in forming and applying the conception of the symbolical addition of arcs of great circles upon a sphere, and in establishing and interpreting an analogous principle or property of such symbolical addition.

As it has been already proposed in the third article of this paper, and also in the works of other writers on subjects connected with the present, to adopt, for the *addition of straight lines having direction*, a rule expressed by the formula

$$\mathrm{CB} + \mathrm{BA} = \mathrm{CA} \qquad (7),$$

in whatever manner the three points ABC may be situated or related to each other; so it seems natural to adopt now, for the analogous *addition of arcs upon a sphere*, when directions as well as lengths are attended to, the corresponding formula,

$$\frown \mathrm{CB} + \frown \mathrm{BA} = \frown \mathrm{CA} \qquad (136).$$

Admitting this latter formula as *the definition of the effect of the sign + when inserted between two such symbols of arcs*, and granting also that it is permitted, in any such formula, to substitute for any arcual symbol another which is *equal* thereto, we shall have, by the two first and two last equations (134) respectively, the two following other equations,

$$\left.\begin{array}{l} \frown \mathrm{B''C''} = \frown \mathrm{AA'} + \frown \mathrm{B'C} \\ \frown \mathrm{A''D''} = \frown \mathrm{AD'} + \frown \mathrm{C'C} \end{array}\right\} \qquad (137).$$

The two sums in these second members will therefore be symbolically equal if we have the equation

$$\frown \mathrm{A'D'} = \frown \mathrm{B'C'} \qquad (138),$$

because (135) has been seen to follow from (133) and (134). But by (136) and (138), we have the expression

$$\frown \mathrm{B'C} = \frown \mathrm{A'D'} + \frown \mathrm{C'C} \qquad (139);$$

consequently the associative principle of multiplication, considered in several recent articles, when combined with the *formula of arcual addition* (136), conducts to the following formula,

$$\frown \mathrm{AA'} + (\frown \mathrm{A'D'} + \frown \mathrm{C'C}) = (\frown \mathrm{AA'} + \frown \mathrm{A'D'}) + \frown \mathrm{C'C} \qquad (140),$$

or, as it may be more concisely written,

$$\frown''' + (\frown'' + \frown') = (\frown''' + \frown'') + \frown' \qquad (141):$$

which in its form agrees with ordinary algebra, and may be said to express the *associative principle of the symbolical addition of arcs*; since the three arcs added in (140) or (141) may be any three arcs of great circles upon one common spheric surface. It is remarkable that so much geometrical meaning should be contained in so simple and elementary a form; for this form (141), which is *apparently an algebraic truism*, and has been here deduced from the associative principle of multiplication of geometric fractions, may reciprocally be substituted for it, and therefore includes in its interpretation, *if we adopt the symbolical definition* (136) of the effect of + between two symbols of arcs, all those theorems respecting spherical great circles, triangles, quadrilaterals, hexagons, and conics, which have been deduced or mentioned as geometrical results of the associative principle in the two foregoing articles. And this encouragement to adopt the foregoing very simple definition (136) of the meaning of a symbol such as $\frown'' + \frown'$, is the more worthy of attention, because the *same definition* conducts to a *departure from the ordinary rules of symbolical addition* in *another* important point; since, when combined with the *definition of symbolical equality between arcs* assigned in the 17th article, it shews that *addition of arcs is in general a non-commutative operation.* For if we conceive two arcs of different great circles on one sphere, from A to B and from C to D, to bisect each other in a point E, we shall then have the two symbolical equations

$$\frown \mathrm{AE} = \frown \mathrm{EB}, \quad \frown \mathrm{CE} = \frown \mathrm{ED} \qquad (142);$$

and therefore, whereas by (136),

$$\frown \mathrm{AE} + \frown \mathrm{ED} = \frown \mathrm{AD} \qquad (143),$$

the result of the addition of the same two arcs, taken in a different order, will be

$$\frown \mathrm{ED} + \frown \mathrm{AE} = \frown \mathrm{CB} \qquad (144).$$

And although the two *sum-arcs*, $\frown$ AD and $\frown$ CB, thus obtained, connecting two opposite pairs of extremities of the two commedial arcs $\frown$ AB and $\frown$ CD, are *equally long*, yet they are in general *parts of different great circles*, and therefore *not symbolically equal* in the sense of the 17th article. This result, which may at first sight seem a paradox, illustrates and is intimately connected with the analogous result obtained in the 13th article, respecting the general non-commutativeness of geometrical multiplication; for we shall find that there exists a species of *logarithmic connexion* between arcs situated in different great circles on a sphere and fractional factors belonging to different planes, which is analogous to, and includes as a limiting case, the known connexion between ordinary imaginary logarithms and angles in a single plane. It may be here remarked, that with the same definition (136) *in any symbolical addition of three successive arcs, the two partial sum-arcs,*

$$\frown'' + \frown' \ \textit{and} \ \frown''' + \frown'' \qquad (145),$$

are portions of the cyclic arcs of a certain spherical conic, circumscribed about a quadrilateral which has

$$\frown', \ \frown'', \ \frown''', \text{ and } \frown''' + \frown'' + \frown' \qquad (146),$$

that is, *the three proposed summand-arcs and their total sum-arc, for portions of its four sides*, or of those sides prolonged; as will appear by supposing that the three summands $\frown'$, $\frown''$, $\frown'''$, coincide respectively with the arcs $\frown$ CC′, $\frown$ B′C, $\frown$ AA′, in the notation of the preceding article.

Symbolical Expressions for a Cyclic Cone; Relations of such a Cone, and of its Cyclic Planes, to a Product of Two Geometrical Fractions.

20. It is evidently a determinate* problem to construct a *cyclic cone*, that is, a cone with circular base (called usually a cone of the second degree), when three of the *sides* (or generating straight lines) of the cone are given in position, and when the plane of the base is parallel to a given *cyclic plane*, which passes through the vertex. To treat this problem, which may be regarded as a fundamental one in the theory of such cones, by a method derived from the principles of the foregoing articles, let the three given sides be denoted by the letters a, b, c; and let the two known lines, in which the given cyclic plane is cut by the planes of the two pairs, ab and bc, be denoted by a′ and b′; also let d denote any fourth side of the sought cyclic cone, and c′, d′ the lines of intersection of the given cyclic plane with the variable planes of cd and da; then, if suitable lengths be assigned to these straight lines, of which the relative *directions* in space are the chief object of the present investigation, the following equality between two products of certain geometrical fractions will exist, and may be regarded as a form of the *equation of the cone*:

$$\frac{\mathrm{c}}{\mathrm{b}'}\frac{\mathrm{a}'}{\mathrm{a}} = \frac{\mathrm{c}}{\mathrm{c}'}\frac{\mathrm{d}'}{\mathrm{a}} \qquad (147).$$

That is to say, when this equation is satisfied, the two lines which are the respective intersections of the planes of the fractional factors of these two equal products, namely the intersection b of the planes aa′ and b′c, and the intersection d of the planes ad′ and c′c, are two sides of a cyclic cone, which has for two other sides the lines a and c, and which has for one cyclic plane the common plane of the four lines a′, b′, c′ and d′; these eight lines a, b, c, d, a′, b′, c′, d′, being here supposed to diverge from one common origin, namely the vertex (or centre) of the cone. This may easily be shown to be a consequence of what has been already established, respecting the connexion of the cyclic arcs of a spherical conic with the symbolic sums of certain other arcs. Or, without introducing any sphere, we may observe that, by (121) and its converse, the equation (147) may be abridged to the following:

$$\frac{\mathrm{a}'}{\mathrm{b}'} = \frac{\mathrm{d}'}{\mathrm{c}'}; \quad \text{or, } \frac{\mathrm{a}'}{\mathrm{b}'}\frac{\mathrm{c}'}{\mathrm{d}'} = 1 \qquad (148);$$

which shows, in virtue of the notation here employed, that besides a certain proportionality of lengths, not necessary now to be considered, there exists an equality between the angles of

* The evident and known determinateness of this problem, corresponding to that of the elementary problem of circumscribing a circle about a given plane trangle, was tacitly assumed, but might with advantage have been expressly referred to, in the outline of a demonstration which was given in the note to Art. (18). The reasoning, towards the end of that note, would then stand thus:—If D be any fourth point on the determined spherical conic, which passes through the three points A, B, C, and has the arc A′B′ for a cyclic arc, it is also a fourth point on the determined spherical conic which passes through the same three points and has the arc B″C″ for a cyclic arc; therefore the two conics, determined by these two sets of conditions, coincide one with the other: or, in other words, the arc B″C″ is a *second* cyclic arc of the *same* spherical conic, of which the arc A′B′ is a *first* cyclic arc.

rotation, in one common plane, which would transport the lines b′ and c′, respectively, into the directions of a′ and d′. But the four lines a′, b′, c′, d′ are respectively parallel to the four symbolic differences, b − a, c − b, d − c, a − d, or to the four straight lines BA, CB, DC, AD, that is to the successive sides of the plane quadrilateral ABCD, if we now suppose the lines a, b, c, d to terminate, in the points A, B, C, D, on a transversal plane parallel to the plane of a′, b′, c′, d′. We may therefore present the relation (148) under either of the two forms:

$$\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}}\frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}} = x'; \quad \text{or,} \ \frac{\mathrm{BA}}{\mathrm{CB}}\frac{\mathrm{DC}}{\mathrm{AD}} = x \qquad (149);$$

in which x is a positive or negative scalar; or, using the characteristic V of the operation of taking the vector part, we may write

$$\mathrm{V} \,.\, \frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}}\frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}} = 0; \quad \text{or } \mathrm{V} \,.\, \frac{\mathrm{BA}}{\mathrm{CB}}\frac{\mathrm{DC}}{\mathrm{AD}} = 0 \qquad (150).$$

When the scalar x is positive, then, by considering the two rotations above mentioned, we may easily perceive that the two points B and D are at one common side of the straight line AC, and that this line subtends equal angles at those two points; being in one common plane with them, as indeed the second equation (149) sufficiently expresses, since it gives

$$\mathrm{V}\frac{\mathrm{BA}}{\mathrm{CB}} = x\mathrm{V}\frac{\mathrm{DA}}{\mathrm{CD}} \qquad (151);$$

so that the two triangles ABC, ADC, on the common base AC, have one common perpendicular to their planes, which must therefore coincide with each other. In the contrary case, namely when x is negative, the equation (151) still shows that the four points are (as above) coplanar with each other; and while the points B and D are now at opposite sides of the line AC, the angles which this line subtends at those two points are now not equal but supplementary. In each case, therefore, the four points ABCD are on the circumference of one common circle; the four lines a, b, c, d are consequently sides of a cyclic cone; and the plane of the four other lines a′, b′, c′, d′ is a cyclic plane of that cone.

21. In the foregoing article, the coplanarity of each of the four sets of three lines, a′ab, b′bc, c′cd, d′da, allows us to suppose that four other lines b″, c″, d″, a″, in the same four planes respectively, and all, like the eight former lines, diverging from the vertex of the cone, are determined so as to satisfy the four equations:

$$\frac{\mathrm{b}''}{\mathrm{b}} = \frac{\mathrm{a}}{\mathrm{a}'}; \quad \frac{\mathrm{c}''}{\mathrm{c}} = \frac{\mathrm{b}}{\mathrm{b}'}; \quad \frac{\mathrm{d}''}{\mathrm{d}} = \frac{\mathrm{c}}{\mathrm{c}'}; \quad \frac{\mathrm{a}''}{\mathrm{a}} = \frac{\mathrm{d}}{\mathrm{d}'} \qquad (152);$$

and then, since these equations, combined with (148), give, by the associative property of the multiplication of geometrical fractions, this other equation,

$$\frac{\mathrm{b}''}{\mathrm{c}''} = \frac{\mathrm{a}''}{\mathrm{d}''} \qquad (153),$$

it follows that these four new lines are in one common plane; and also that the rotations in that plane, from b″ and c″ to a″ and d″, respectively, are equal. And this new plane is evidently a *second* cyclic plane of the same cone*; for we may now write, instead of (147), the analogous equation:

$$\frac{\mathrm{c}}{\mathrm{c}''}\frac{\mathrm{b}''}{\mathrm{a}} = \frac{\mathrm{c}}{\mathrm{d}''}\frac{\mathrm{a}''}{\mathrm{a}} \qquad (154);$$

the two members being here equal respectively to the reciprocals of the two members of the first equation (148): nor is it necessary to retain the restriction that the lines a, b, c, d should terminate in one common plane. In like manner, the two members of the equation (147) are respectively equal to the reciprocals of the two members of the equation (153); a geometrical (like an arithmetical) fraction being said to be changed to its *reciprocal*, when the numerator and denominator are interchanged. We have therefore this theorem:—*A cyclic cone is the locus of the intersection of the planes of two geometrical fractions, of which the product is a constant fraction, while the numerator of the multiplier and the denominator of the multiplicand are constant lines. These two lines are two fixed sides of the cone; the plane of the two other and variable lines*, which enter as denominator and numerator into the expressions of the same two fractional factors, *is one cyclic plane of that cone*; and *the plane of the constant product is the other cyclic plane.* The investigation in the last article shows also that the condition for four points ABCD being *concircular* or *homocyclic*, that is, for their being corners of a quadrilateral inscribed in a circle, is expressed by the second equation (150); which may therefore be called the *equation of homocyclicism.* The same investigation shows that if we only know that ABCD are four points on one common plane, we may still write an equation of the form (151); which may for that reason be said to be a *formula of coplanarity.*

Symbolical Expressions and Investigations of some Properties of Cyclic Cones, with reference to their Tangent Planes.

22. If the side b of the cyclic cone be conceived to approach to the side a, and ultimately to coincide with it, the first equation (152) will take this limiting form:

$$\frac{\mathrm{b}''}{\mathrm{a}} = \frac{\mathrm{a}}{\mathrm{a}'} \qquad (155);$$

which expresses the known theorem that the side of contact a bisects the angle between the traces a′ and b″ of the tangent plane on the two cyclic planes; bisecting also the vertically opposite angle between the traces −a′ and −b″, but being perpendicular to the bisector of either of the two other angles, which are supplementary to the two already mentioned, namely the angle between the traces a′ and −b″, and that between −a′ and b″. And if in like manner we conceive the side d to approach indefinitely to the side c, the plane of these two sides will tend to become another tangent plane to the cone; of which plane the traces c′ and d″ on the two cyclic planes will satisfy an equation of the same form as that last written, namely the following, which is the limiting form of the third equation (152):

$$\frac{\mathrm{d}''}{\mathrm{c}} = \frac{\mathrm{c}}{\mathrm{c}'} \qquad (156).$$

* See the remarks made in the note to the foregoing article.

At the same time, the two secant planes bc and da will tend to coalesce in one secant plane, containing the two sides of contact a and c, with which the two other sides b and d tend to coincide; so that the traces d′ and a″ of the latter secant plane, on the two cyclic planes, will ultimately coincide with the traces b′ and c″ of the former secant plane on the same two cyclic planes; and the equations (148) (153) become:

$$\frac{\mathrm{a}'}{\mathrm{b}'} = \frac{\mathrm{b}'}{\mathrm{c}'}; \quad \frac{\mathrm{b}''}{\mathrm{c}''} = \frac{\mathrm{c}''}{\mathrm{d}''} \qquad (157);$$

which expresses that the traces b′ and c″ of the one remaining secant plane bisect respectively the angles between the pairs of traces, a′, c′, and b″, d″, of the two tangent planes, on the two cyclic planes. And the two remaining equations (152) concur in giving this other equation:

$$\frac{\mathrm{c}''}{\mathrm{c}} = \frac{\mathrm{a}}{\mathrm{b}'} \qquad (158):$$

expressing that the rotations in the secant plane from b′ to a and from c to c″, that is to say from one trace to one side, and from the other side to the other trace, are equal in amount, and similarly directed; in such a manner that these two traces b′ and c″, of the secant plane on the two cyclic planes, are equally inclined to the straight line which bisects the angle between these two sides a and c, along which the plane cuts the cone: all of which agrees with the known properties of cones of the second degree.

23. The eight straight lines a, c, a′, b′, c′, b″, c″, d″, being supposed to be equally long, the first of them, which has been seen to coincide in direction with the bisector of the angle between the third and sixth, can differ only by a scalar (or real and numerical) coefficient from their symbolic sum; because the diagonals of a plane and equilateral quadrilateral figure (or rhombus) bisect the angles of that figure. We have therefore, by (155), and by the supposition of the equal lengths of the eight lines,

$$\mathrm{a}' + \mathrm{b}'' \parallel \mathrm{a}; \quad \text{or,} \quad \mathrm{a}' + \mathrm{b}'' = l\mathrm{a} \qquad (159),$$

l being a numerical coefficient, and the sign of parallelism being designed to include the case of coincidence.

In like manner, by (156), we have

$$\mathrm{d}'' + \mathrm{c}' \parallel \mathrm{c}; \quad \text{or,} \quad \mathrm{d}'' + \mathrm{c}' = l'\mathrm{c} \qquad (160),$$

l' being another scalar coefficient. Again, by (157),

$$\left.\begin{array}{ll} \mathrm{c}' + \mathrm{a}' \parallel \mathrm{b}'; & \mathrm{c}' + \mathrm{a}' = m\mathrm{b}'; \\ \mathrm{b}'' + \mathrm{d}'' \parallel \mathrm{c}''; & \mathrm{b}'' + \mathrm{d}'' = m'\mathrm{c}''; \end{array}\right\} \qquad (161),$$

m and m' being two other scalars. But, by (158),

$$\frac{\mathrm{c}''}{\mathrm{c}}\frac{\mathrm{b}'}{\mathrm{a}} = 1 \qquad (162);$$

therefore

$$\frac{\mathrm{b}'' + \mathrm{d}''}{\mathrm{d}'' + \mathrm{c}'} \frac{\mathrm{c}' + \mathrm{a}'}{\mathrm{a}' + \mathrm{b}''} = \frac{m'}{l'} \frac{m}{l} = \mathrm{V}^{-1}0 \qquad (163);$$

this symbol $\mathrm{V}^{-1}0$ denoting generally, in the present system, *any geometrical fraction of which the vector part is zero*, and therefore any positive or negative number (including zero). (Compare the definition and remarks in the 7th article).

By comparing this equation (163) with the first form (150), we see that the four straight lines,

$$-\mathrm{b}'', \ \mathrm{d}'', \ -\mathrm{c}', \ \mathrm{a}' \qquad (164),$$

which have been supposed to diverge from one common origin, namely the vertex of the cone, have their terminations on the circumference of one common circle. But these four lines, by supposition, are also equally long; they must therefore be four sides of a new cone, which is not only cyclic, as having a circular base, but is also a *cone of revolution.* The axis of revolution of this new cone is perpendicular to the plane of the circle in which the four lines (164) terminate; and this plane is parallel to the plane of the symbolic differences of those four lines, namely, the following,

$$\mathrm{d}'' + \mathrm{b}'', \ -\mathrm{c}' - \mathrm{d}'', \ \mathrm{a}' + \mathrm{c}', \ -\mathrm{b}'' - \mathrm{a}' \qquad (165);$$

but these have been seen to be parallel respectively to the four lines c'', c, b', a, which are contained in the secant plane of the former cone; consequently the axis of revolution of the new cone is perpendicular to this secant plane. We arrive therefore, by this symbolical process, at a new proof of the known theorem, discovered by M. Chasles,* that two planes, touching a cyclic cone along any two sides, intersect the two common cyclic planes in four right lines, which are sides of one common cone of revolution, whose axis of revolution is perpendicular to the plane of the two sides of contact of the former cone.

24. If we conceive the first and fourth of the sides (164) of the cone of revolution to tend to coincide with each other, then the fourth of the sides (165) of the plane quadrilateral inscribed in the circular base of that cone will tend to vanish; consequently the direction of this last mentioned side $-\mathrm{b}'' - \mathrm{a}'$, or the opposite direction of $\mathrm{a}' + \mathrm{b}''$, will become at last tangential to this circular base; and the plane of the two sides previously mentioned, namely $-\mathrm{b}''$ and a', which plane has been seen to touch the cyclic cone along the side a, will become ultimately tangential also to the cone of revolution, touching it along the line a', which becomes one trace of the second cyclic plane on the first cyclic plane; the opposite line, $-\mathrm{a}'$, being of course also situated in the intersection of those two planes, so that it may be regarded as the opposite trace of one cyclic plane on the other. Thus, at the limit here considered, the equation (155) and the second equation (157) are replaced by the equations

$$\frac{-\mathrm{a}'}{\mathrm{a}} = \frac{\mathrm{a}}{\mathrm{a}'}, \quad \frac{-\mathrm{a}'}{\mathrm{c}''} = \frac{\mathrm{c}''}{\mathrm{d}''} \qquad (166);$$

* See the Translation of Two General Memoirs by M. Chasles, on the General Properties of Cones of the Second Degree, and on the Spherical Conics; which Translation was published, with an Appendix, by the Rev. Charles Graves, in Dublin, 1841.

of which the first expresses that the side a is equally inclined to the two opposite traces, a' and $-\mathrm{a}'$; while the numerical coefficient l vanishes, and the formula (163) is replaced by this other,

$$\mathrm{V} \,.\, \frac{\mathrm{d}'' - \mathrm{a}'}{\mathrm{d}'' + \mathrm{c}'} \frac{\mathrm{c}' + \mathrm{a}'}{\mathrm{a}} = 0 \qquad (167).$$

We see also that the two rectangular but equally long lines a, a', of which the former is a side of the cyclic cone, while the latter is part of the line of intersection of the two cyclic planes of that cone, are such that their plane is a common tangent to both the cyclic cone and the cone of revolution; which latter cone has also, as sides of the same sheet with a', the two other of the four lines (164), namely the lines $-\mathrm{c}'$ and d''. Indeed, the formula (167) is sufficient to show, by comparison with the first formula (150), that if the three straight lines a', d'', $-\mathrm{c}'$ be still supposed to diverge from one common origin, the circle passing through the three points in which they terminate is touched, at the termination of the line a', by a straight line parallel to the line a; and therefore that the cone of revolution, having these three equally long lines a', $-\mathrm{c}'$, d'' for sides of one common sheet, is touched along the side a' by the plane which contains the two rectangular lines aa'; so that we may regard this formula (167) as containing the symbolical solution to the problem, to draw a tangent plane, along any proposed side, to the cone of revolution which passes through that side and through two other sides also given, and belonging to the same sheet as the former. Now if three such sides be connected by three planes, forming three faces of a triangular pyramid, inscribed in a single sheet of a cone of revolution, and having its vertex at the vertex of that cone, while the sheet is touched by a fourth plane along one edge of the pyramid, it follows from the most elementary principles of solid geometry, that the difference between the two exterior angles which the faces meeting at that edge make with the tangent plane to the cone is equal to the difference of the two interior angles which the same two faces make with the third face of the pyramid; the greater exterior angle being the one which is the more remote from the greater interior angle; as may be shown by conceiving three planes to pass through the three edges respectively, and through the axis of revolution of the cone. The same equality between the differences of these two pairs of angles between planes, will become still more evident if, without making use of any formula of spherical trigonometry, we consider a spherical triangle inscribed in a small circle on the sphere, which small circle is touched at one corner of the triangle by a great circle, while arcs are drawn to that and to the two other corners from a pole of the small circle; the only principles required being these: that the base angles of a spherical isosceles triangle are equal, and that the arcs from the pole of a small circle are all perpendicular to its perimeter. If then we denote by the symbol $\angle(\mathrm{a}, \mathrm{b}, \mathrm{c})$ the acute or right or obtuse dihedral or spherical angle, at the edge b, between the planes ab and bc, in such a manner as to write, generally,

$$\angle(\mathrm{a}, \mathrm{b}, \mathrm{c}) = \angle(\mathrm{c}, \mathrm{b}, \mathrm{a}) = \angle(-\mathrm{a}, \mathrm{b}, -\mathrm{c}) = \angle(\mathrm{a}, -\mathrm{b}, \mathrm{c}) = \pi - \angle(\mathrm{a}, \mathrm{b}, -\mathrm{c}) \qquad (168);$$

π being the symbol for two right angles, we shall have, in the present question, the equation

$$\angle(\mathrm{a}', \mathrm{d}'', -\mathrm{c}') - \angle(\mathrm{a}', -\mathrm{c}', \mathrm{d}'') = \angle(-\mathrm{a}, \mathrm{a}', -\mathrm{c}') - \angle(\mathrm{a}, \mathrm{a}', \mathrm{d}'') \qquad (169);$$

and therefore, by subtracting both members from π,

$$\angle(\mathrm{a}', \mathrm{d}'', \mathrm{c}') + \angle(\mathrm{a}', \mathrm{c}', \mathrm{d}'') = \angle(-\mathrm{a}, \mathrm{a}', \mathrm{c}') + \angle(\mathrm{a}, \mathrm{a}', \mathrm{d}'') \qquad (170).$$

We have also here the relation

$$\angle(\mathrm{c}', \mathrm{a}', \mathrm{d}'') = \angle(\mathrm{a}, \mathrm{a}', \mathrm{c}') + \angle(\mathrm{a}, \mathrm{a}', \mathrm{d}'') \qquad (171),$$

because the plane aa' is intermediate between the planes $\mathrm{a}'\mathrm{c}'$ and $\mathrm{a}'\mathrm{d}''$, or lies *within* the dihedral angle $(\mathrm{c}', \mathrm{a}', \mathrm{d}'')$ itself, and not within either of the two angles which are exterior and supplementary thereto; which again depends on the circumstance that both the cyclic planes are necessarily exterior to each sheet of the cyclic cone. Adding therefore the equations (170) and (171), member to member, and subtracting π on both sides of the result, we find for the *spherical excess* of the new triangular pyramid $(\mathrm{a}', \mathrm{c}', \mathrm{d}'')$, or for the excess of the sum of the mutual inclinations of its three faces $\mathrm{a}'\mathrm{c}'$, $\mathrm{a}'\mathrm{d}''$, $\mathrm{c}'\mathrm{d}''$, above two right angles, the expression:

$$\angle(\mathrm{a}', \mathrm{d}'', \mathrm{c}') + \angle(\mathrm{a}', \mathrm{c}', \mathrm{d}'') + \angle(\mathrm{c}', \mathrm{a}', \mathrm{d}'') - \pi = 2\angle(\mathrm{a}, \mathrm{a}', \mathrm{d}'') \qquad (172).$$

This spherical excess therefore remains unchanged, while the two lines c', d'', move together on the two cyclic planes, in such a manner that their plane, always passing through the vertex of the cone, continues to touch that cyclic cone; a' being still a line situated in the intersection of the two cyclic planes, and a being still a side of contact of the cone with a plane drawn through that intersection. And hence, or more immediately from the equation (170), the known property of a cyclic cone is proved anew, that the sum of the inclinations (suitably measured) of its variable tangent plane to its two fixed cyclic planes is constant.

Condition of Concircularity, resumed. New Equation of a Cyclic Cone.

25. The equation (150) of *homocyclism*, or of *concircularity*, which was assigned in the 20th article, and which expresses the condition requisite in order that four straight lines in space, a, b, c, d, diverging from one common point O, as from an origin, may terminate in four other points A, B, C, D, which shall all be contained on the circumference of one common circle, may also, by (149), be put under the form

$$\frac{\mathrm{b} - \mathrm{a}}{\mathrm{c} - \mathrm{b}} = x\frac{\mathrm{d} - \mathrm{a}}{\mathrm{c} - \mathrm{d}} \qquad (173),$$

where x is a scalar coefficient. It gives therefore the two following separate equations, one between scalars, and the other between vectors:

$$\mathrm{S}\frac{\mathrm{b} - \mathrm{a}}{\mathrm{c} - \mathrm{b}} = x\mathrm{S}\frac{\mathrm{d} - \mathrm{a}}{\mathrm{c} - \mathrm{d}}; \quad \mathrm{V}\frac{\mathrm{b} - \mathrm{a}}{\mathrm{c} - \mathrm{b}} = x\mathrm{V}\frac{\mathrm{d} - \mathrm{a}}{\mathrm{c} - \mathrm{d}} \qquad (174);$$

of which the latter is only another way of writing the equation (151). If then we agree to use, for conciseness, a new characteristic of operation, $\frac{\mathrm{V}}{\mathrm{S}}$, of which the effect on any geometrical fraction, to the symbol of which it it prefixed, shall be defined by the formula

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{b}}{\mathrm{a}} = \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}} \div \mathrm{S}\frac{\mathrm{b}}{\mathrm{a}} \qquad (175);$$

so that this new characteristic $\frac{\mathrm{V}}{\mathrm{S}}$, which (it must be observed) *is not a distributive symbol*, is to be considered as directing to *divide the vector by the scalar part* of the geometrical fraction on which it operates; we shall then have, as a consequence of (173), this other form of the *equation of concircularity*:

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = \frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \qquad (176).$$

Conversely we can return from this latter form (176) to the equation (173); for if we observe that, in the present system of symbolical geometry, *every geometrical fraction is equal to the sum of its own scalar and vector parts*, so that we may write generally (see article 7),

$$\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}} + \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}} = \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}} + \mathrm{S}\frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{b}}{\mathrm{a}} \qquad (177),$$

or more concisely,

$$\mathrm{S} + \mathrm{V} = \mathrm{V} + \mathrm{S} = 1 \qquad (178);$$

and, if we add the identity,

$$\mathrm{S}\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} \div \mathrm{S}\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = \mathrm{S}\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \div \mathrm{S}\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \qquad (179),$$

of which each member is unity, to the equation (176), attending to the definition (175) of the new characteristic lately introduced, we are conducted to this other formula,

$$\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} \div \mathrm{S}\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} = \frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \div \mathrm{S}\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \qquad (180);$$

which allows us to write also

$$\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} \div \frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} = \mathrm{S}\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}} \div \mathrm{S}\frac{\mathrm{d}-\mathrm{a}}{\mathrm{c}-\mathrm{d}} \qquad (181),$$

where the second member, being the quotient of two scalars, is itself another scalar, which may be denoted by x; and thus the equation (173) may be obtained anew, as a consequence of the equation (176). We may therefore also deduce from the last-mentioned equation the following form,

$$\frac{\mathrm{c}-\mathrm{d}}{\mathrm{d}-\mathrm{a}} = x\frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}} \qquad (182);$$

and thence also, by a new elimination of the scalar coefficient x, performed in the same manner as before, may derive this other form,

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{c}-\mathrm{d}}{\mathrm{d}-\mathrm{a}} = \frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}} \qquad (183).$$

Indeed, the geometrical signification of the condition (176) shews easily that we may in any manner transpose, in that condition, the symbols a, b, c, d; since if, *before* such a transposition, those symbols denoted four diverging straight lines (not generally in one common plane), which terminate on the circumference of one common circle, then *after* this transposition they must still denote four such diverging lines. We may therefore interchange the symbols a and c, in the condition (176), which will thus become

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{b}-\mathrm{c}}{\mathrm{a}-\mathrm{b}} = \frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}} \tag{184}$$

but also, as in ordinary algebra, we have here,

$$\frac{\mathrm{b}-\mathrm{c}}{\mathrm{a}-\mathrm{b}} = \frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}}; \quad \frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}} = \frac{\mathrm{c}-\mathrm{d}}{\mathrm{d}-\mathrm{a}} \tag{185}$$

the equation (183) might therefore have been in this other way deduced from the equation (176), as another form of the same condition of concircularity: and it is obvious that several other forms of the same condition may be obtained in a similar way.

26. From the fundamental importance of the *circle* in geometry, it is easy to foresee that these various forms of the condition of concircularity must admit of a great number of geometrical applications, besides those which have already been given in some of the preceding articles of this essay on Symbolical Geometry. For example, we may derive in a new way a solution of the problem proposed at the beginning of the 20th article, by conceiving that the symbols a, b, c denote three given sides of a *cyclic cone*, extending from the vertex to some given plane which is parallel to that one of the two *cyclic planes* which in the problem is supposed to be given; for then the equation (183) may be employed to express that the variable line d is a fourth side of the same cyclic cone, drawn from the same vertex as an origin, and bounded by the same given plane, or terminating on the same circumference, or circular base of the cone, as the three given sides, a, b, c. Or we may change the symbol d to another symbol of the form $x\mathrm{x}$, and may conceive that x denotes a variable side of the cone, still drawn as before from the vertex, but not now terminating on any one fixed plane, nor otherwise restricted as to its length; while x shall denote a scalar coefficient or multiplier, so varying with the side or line x as to render the product-line $x\mathrm{x}$ a side of which the extremity is (like that of d) concircular with the gven extremities of a, b, c; and we may express these conceptions and conditions by writing as the equation of the cone the following:

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{c}-x\mathrm{x}}{x\mathrm{x}-\mathrm{a}} = \beta \tag{186}$$

where β is a given geometrical fraction of the *vector* class, namely, that vector which is determined by the equation

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}} = \beta \tag{187}$$

The *index* $\mathrm{I}\beta$ of this vector β is such that

$$\mathrm{I}\beta \parallel \mathrm{I}\frac{\mathrm{c}-\mathrm{b}}{\mathrm{b}-\mathrm{a}} \tag{188}$$

it is therefore (by the principles of articles 7 and 10) a line perpendicular to each of the two lines represented by the two symbolical differences $\mathrm{c}-\mathrm{b}$, $\mathrm{b}-\mathrm{a}$, and therefore also perpendicular to the line denoted by their symbolical sum, $\mathrm{c}-\mathrm{a}$; so that we may establish the three formulæ,

$$\mathrm{I}\beta \perp \mathrm{c}-\mathrm{b}; \quad \mathrm{I}\beta \perp \mathrm{b}-\mathrm{a}; \quad \mathrm{I}\beta \perp \mathrm{c}-\mathrm{a} \tag{189}$$

and may say that $\mathrm{I}\beta$ is a line *perpendicular to the plane* in which all the three lines a, b, c all terminate. This constant index $\mathrm{I}\beta$, connected with the equation (186) of the cyclic cone just now determined, as being the *index of the constant vector fraction* β, to which the first member of that equation is equal, is therefore perpendicular also to the given cyclic plane of the same cone, and may be regarded as a symbol for one of the two *cyclic normals* of that conical locus of the variable line x lately considered. In the particular case when the three given lines a, b, c are all *equally long*, so that the cyclic cone (186) becomes a *cone of revolution*, then the index $\mathrm{I}\beta$, which had been generally a symbol for a cyclic normal, becomes a symbol for the *axis of revolution* of the cone. Other forms of equations of such cyclic and other cones will offer themselves when the principles of the present system of symbolical geometry shall have been more completely unfolded; but the forms just given will be found to be sufficient, when combined with some of the equations assigned in previous articles, to conduct to the solution of some interesting geometrical problems: to which class it will perhaps be permitted to refer the general determination of the *curvature of a spherical conic*, or the construction of the cone of revolution which *osculates* along a given side to a given cyclic cone.

Curvature of a Spherical Conic, or of a Cyclic Cone.

27. To treat this problem by a method which shall harmonise with the investigations of recent articles of this paper, let the symbols a', c', d'', be employed with the same significations as in article 24, so as to denote three equally long straight lines, of which a' is a trace of one cyclic plane on the other, while c' and d'' are the traces of a tangent plane on those two cyclic planes; and let c (still bisecting the angle between c' and d'') be still the equally long side of contact of that tangent plane with the given cyclic cone. We shall then have, by (156), the symbolic analogy,

$$\mathrm{d}'' : \mathrm{c} :: \mathrm{c} : \mathrm{c}' \tag{190}$$

which, on account of the supposed equality of the lengths of the lines c, c', d'', gives also the two following formulæ, of parallelism and perpendicularity,

$$\mathrm{d}'' + \mathrm{c}' \parallel \mathrm{c}; \quad \mathrm{d}'' - \mathrm{c}' \perp \mathrm{c}; \tag{191}$$

of which indeed the former has been given already, as the first of the two formulæ (160). Conceive next that through the side of contact c we draw two secant planes, cutting the same sheet of the cone again in two known sides c_1, c_2, and having for their known traces on the first cyclic plane (which contains the trace c' of the tangent plane) the lines c'_1, c'_2, but for their traces on the second cyclic plane (or on that which contains d'') the lines d''_1, d''_2; these lines, $\mathrm{c}\mathrm{c}_1\mathrm{c}_2\mathrm{c}'_1\mathrm{c}'_2\mathrm{d}''_1\mathrm{d}''_2$, being supposed to be all equally long. We may then write (in virtue of what has been shewn in former articles) at once the two new symbolic *analogies*,

$$\mathrm{d}''_1 : \mathrm{c}_1 :: \mathrm{c} : \mathrm{c}'_1; \quad \mathrm{d}''_2 : \mathrm{c}_2 :: \mathrm{c} : \mathrm{c}'_2 \tag{192}$$

the two new *parallelisms*,

$$\mathrm{d}_1'' + \mathrm{c}_1' \parallel \mathrm{c}_1 + c; \quad \mathrm{d}_2'' + \mathrm{c}_2' \parallel \mathrm{c}_2 + c \tag{193;}$$

and the two new *perpendicularities*,

$$\mathrm{d}_1'' - \mathrm{c}_1' \perp \mathrm{c}_1 + c; \quad \mathrm{d}_2'' - \mathrm{c}_2' \perp \mathrm{c}_2 + c \tag{194 :}$$

we shall have also these two other formulæ of parallelism,

$$\mathrm{d}_1'' - \mathrm{c}_1' \parallel \mathrm{c}_1 - c; \quad \mathrm{d}_2'' - \mathrm{c}_2' \parallel \mathrm{c}_2 - c \tag{195.}$$

Now if we conceive a cone of revolution to contain upon one sheet the three equally long lines c, c_1 c_2, which are also (by the construction) three sides of one sheet of the given cyclic cone, we may (by the last article) represent a line in the direction of the *axis* of this cone of revolution by the symbol,

$$\mathrm{I}\frac{\mathrm{c}_2 - \mathrm{c}}{\mathrm{c} - \mathrm{c}_1} \tag{196;}$$

or by this other symbol, which denotes indeed a line having an opposite direction, but still one contained upon the indefinite axis of the same cone of revolution, if drawn from a point on that axis,

$$\mathrm{I}\frac{\mathrm{c}_2 - \mathrm{c}}{\mathrm{c}_1 - \mathrm{c}} \tag{197.}$$

On account of the parallelisms (195) we may substitute for the last symbol (197) this other of the same kind,

$$\mathrm{I}\frac{\mathrm{d}_2'' - \mathrm{c}_2'}{\mathrm{d}_1'' - \mathrm{c}_1'} \tag{198;}$$

which expression, when we add to it another, which is a symbol of a null line (because in general the index of a scalar vanishes), namely the following,

$$\mathrm{I}\frac{\mathrm{c}_1' - \mathrm{d}_1''}{\mathrm{d}_1'' - \mathrm{c}_1'} = 0 \tag{199,}$$

takes easily this other form,

$$\mathrm{I}\frac{\mathrm{d}_2'' - \mathrm{c}_2'}{\mathrm{d}_1'' - \mathrm{c}_1'} = \mathrm{I}\frac{\mathrm{d}_2'' - \mathrm{d}_1''}{\mathrm{d}_1'' - \mathrm{c}_1'} + \mathrm{I}\frac{\mathrm{c}_1' - \mathrm{c}_2'}{\mathrm{d}_1'' - \mathrm{c}_1'} \tag{200.}$$

The sought axis of the cone of revolution through the sides $\mathrm{cc}_1\mathrm{c}_2$ of the cyclic cone, or a line in the direction of this axis, is therefore thus given, by the expression (200), as the symbolic *sum* of two other lines; which two new lines, by comparison of their expressions with the form (188), are seen to be in the directions of the axes of revolution of two new or *auxiliary cones* of revolution; one of these auxiliary cones containing, upon a single sheet, the three lines

$$\mathrm{c}_1', \ \mathrm{d}_1'', \ \mathrm{d}_2'' \tag{201,}$$

so that it may be briefly called the cone of revolution $c_1' \; d_1'' \; d_2''$; while the other auxiliary cone of revolution, which may be called in like manner the cone $c_2' \; c_1' \; d_1''$, contains on one sheet this other system of three straight lines,

$$c_1', \;\; d_1'', \;\; c_2' \tag{202}$$

The symbolic *difference* of the same two lines, namely, that of the lines denoted by the symbols

$$\mathrm{I}\frac{d_2'' - d_1''}{d_1'' - c_1'}, \quad \mathrm{I}\frac{c_1' - c_2'}{d_1'' - c_1'} \tag{203}$$

which lines are thus in the directions of the axes of these two new cones of revolution, may easily be expressed under the form

$$\mathrm{I}\frac{\frac{1}{2}(d_2'' + c_2') - \frac{1}{2}(d_1'' + c_1')}{\frac{1}{2}(d_1'' + c_1') - c_1'} \tag{204}$$

it is therefore (by the same last article) a line perpendicular to the plane in which the three following straight lines terminate, if drawn from one common point, such as the common vertex of the four cones,

$$c_1', \quad \tfrac{1}{2}(d_1'' + c_1'), \quad \tfrac{1}{2}(d_2'' + c_2') \tag{205}$$

This plane contains also the termination of the line d_1'', if that line be still drawn from the same vertex; because, in general, whatever may be the value of the scalar x, the three straight lines denoted by the symbols

$$c_1', \quad (1 - x)d_1'' + xc_1', \quad d_1'' \tag{206}$$

all terminate on one straight line, if they be drawn from one common origin; and this last straight line is situated in the first secant plane, and connects the extremities of the two equally long lines c_1', d_1'', which are the traces of that second plane on the two cyclic planes. The remaining line, $\frac{1}{2}(d_2'' + c_2')$, of the system (205), if still drawn from the same vertex as before, bisects that other straight line, situated in the same secant plane, which connects the extremities of the two equally long traces c_2', d_2'', of that other secant plane on the same two cyclic planes. And these two connecting lines, thus situated respectively in the first and second secant planes do not generally intersect each other; because they cut the line of mutual intersection of those two secant planes, namely the side c of the given cyclic cone, in points which are in general situated at different distances from the vertex. It is therefore in general a determinate problem, to draw through the first of these two connecting lines a plane which shall bisect the second: and we see that the plane so drawn, being that in which the three lines (205) terminate, is perpendicular to the line (204), that is to the symbolic difference,

$$\mathrm{I}\frac{d_2'' - d_1''}{d_1'' - c_1'} - \mathrm{I}\frac{c_1' - c_2'}{d_1'' - c_1'} \tag{207}$$

of the two lines (203), of which the symbolic sum (200) has been seen to be a line in the direction of the axis (197) of the first cone of revolution considered in the present article;

while the two lines (203), of which we have thus taken the symbolic sum and difference, have been perceived to be in the directions of the axes of the two other and auxiliary cones of revolution, which we have also had occasion to consider. But in general, by one of those fundamental principles which the present system of symbolical geometry has in *common* with other systems, the symbolical sum and difference of two adjacent and coinitial sides of a parallelogram may be represented or constructed geometrically by the two diagonals of that figure; namely the sum by that diagonal which is intermediate between the two sides, and the difference by that other diagonal which is transversal to those sides: and every other transversal straight line, which is drawn across the same two sides in the same direction as the second diagonal, is bisected by the first diagonal, because the two diagonals themselves bisect each other. We may therefore enunciate this theorem:—*If across the axes* (203) *of the two auxiliary cones of revolution, which contain respectively the two systems of straight lines* (201) *and* (202), (each system of three straight lines being contained upon a single sheet), *we draw a rectilinear transversal, perpendicular to the plane which contains the first and bisects the second of the two connecting lines, drawn as before in the two secant planes; and if we then bisect this transversal by a straight line drawn from the common vertex of the cones: this bisecting line will be situated on the axis of revolution* (197) *of that other cone of revolution, which contains upon one sheet the three given sides of the given cyclic cone.* (The drawing of this transversal is possible, because the preceding investigation shews that the plane of the axes of revolution of the two auxiliary cones is perpendicular to that other plane which is described in the construction.)

28. Since, generally, in the present system of symbolical geometry, the vector part of the quotient of any two parallel lines, and the scalar part of the quotient of any two perpendicular lines, are respectively equal to zero, we may express that *three* straight lines a, b, c, if drawn from a common origin, all *terminate on one common straight line*, by writing the equation

$$\mathrm{V}\frac{\mathrm{c}-\mathrm{a}}{\mathrm{b}-\mathrm{a}} = 0 \qquad (208);$$

and may express that *two* straight lines, a, c, are *equally long*, or that they are fit to be made adjacent sides of a rhombus (of which the two diagonals are mutually rectangular), by this other formula:

$$\mathrm{S}\frac{\mathrm{c}+\mathrm{a}}{\mathrm{c}-\mathrm{a}} = 0 \qquad (209).$$

If then we combine these two conditions, which will give

$$\mathrm{S}\frac{\mathrm{c}+\mathrm{a}}{\mathrm{b}-\mathrm{a}} = 0 \qquad (210),$$

and therefore

$$\mathrm{S}\frac{\mathrm{c}}{\mathrm{b}-\mathrm{a}} = -\mathrm{S}\frac{\mathrm{a}}{\mathrm{b}-\mathrm{a}}, \quad \mathrm{V}\frac{\mathrm{c}}{\mathrm{b}-\mathrm{a}} = \mathrm{V}\frac{\mathrm{a}}{\mathrm{b}-\mathrm{a}} \qquad (211),$$

we shall thereby express that the chord or secant of a circle or sphere, which passes through the extremity of one given radius a, and also through the extremity of another given and coinitial straight line b, meets the circumference of the same circle or the surface of the same sphere again at the extremity of the other straight line denoted by c, which will thus be another radius. But with the same mode of abridgment as that employed in the formula (178), we have, by (211),

$$(\mathrm{V}+\mathrm{S})\frac{\mathrm{c}}{\mathrm{b}-\mathrm{a}} = (\mathrm{V}-\mathrm{S})\frac{\mathrm{a}}{\mathrm{b}-\mathrm{a}} \qquad (212),$$

and therefore

$$\mathrm{c} = (\mathrm{V}-\mathrm{S})\frac{\mathrm{a}}{\mathrm{b}-\mathrm{a}} \,.\, (\mathrm{b}-\mathrm{a}) \qquad (213).$$

This last is consequently an expression for the second radius c, in terms of the first radius a, and of the other given line b from the same centre, which terminates at some given point upon the common chord or secant, connecting the extremities of the two radii. If therefore we write for abridgment

$$\mathrm{m} = \tfrac{1}{2}(\mathrm{d}_2'' + \mathrm{c}_2') \qquad (214),$$

so that m shall be a symbol for the last of the three lines (205); and if we employ the two following expressions, formed on the plan (213),

$$\left.\begin{aligned} \mathrm{m}' &= (\mathrm{V}-\mathrm{S})\frac{\mathrm{c}_1'}{\mathrm{m}-\mathrm{c}_1'} \,.\, (\mathrm{m}-\mathrm{c}_1') \\ \mathrm{m}'' &= (\mathrm{V}-\mathrm{S})\frac{\mathrm{d}_1''}{\mathrm{m}-\mathrm{d}_1''} \,.\, (\mathrm{m}-\mathrm{d}_1'') \end{aligned}\right\} \qquad (215),$$

the symbols c_1', d_1'' retaining their recent meanings; then the four straight lines,

$$\mathrm{c}_1',\ \ \mathrm{d}_1'',\ \ \mathrm{m}',\ \ \mathrm{m}'' \qquad (216),$$

all drawn from the given vertex of the cones, will be equally long, and will terminate in four concircular points; or, in other words, their extremities will be the four corners of a certain quadrilateral inscribed in a circle: of which plane quadrilateral the two diagonals, connecting respectively the ends of c_1', m', and of d_1'', m'', will intersect each other at the extremity of the line m, which is drawn from the same vertex as before. It may also be observed respecting this line m, that in virtue of its definition (214), and of the second parallelism (193), it bisects the angle between the two equally long sides c, c_2 of the given cyclic cone. Thus *the four lines* (216) *are four sides of one common sheet of a new cone of revolution, of which the axis is perpendicular to the plane described in the construction of the foregoing article*; because these four equally long lines (216) terminate on the same plane as the three lines (205), that is on a plane perpendicular to the line (204) or (207), which latter line has thus the direction of the axis of revolution of the new auxiliary cone. It is usual to say that four diverging straight lines are *rays of an harmonic pencil*, or simply that they are *harmonicals*, when a rectilinear transversal, parallel to the fourth, and bounded by the first and third, is bisected by the second of these lines: so that, in general, any four diverging straight lines which can be represented by the four symbols

$$\mathrm{a},\ \ \mathrm{a}+\mathrm{b},\ \ \mathrm{b},\ \ \mathrm{a}-\mathrm{b},$$

or by the symbols which are obtained from these by giving them any scalar coefficients, have the *directions* of four such harmonicals. We are then entitled to assert that *the fourth harmonical to the axes of the three cones of revolution*

$$(c'_1 d''_1 d''_2), \ (c c_1 c_2), \ (c'_2 c'_1 d''_1) \qquad (217),$$

which three axes have been already seen to be all situated in one common plane, *is the axis of that new or fourth cone of revolution* $(c'_1 d''_1 m' m'')$, *which contains on one sheet the four straight lines* (216). And if we regard the four last-mentioned lines as *edges of a tetrahedral angle*, inscribed in this new cone of revolution, we see that *the two diagonal planes* of this tetrahedral angle *intersect each other along a straight line* m, *which bisects the plane angle* (c, c_2) *between two of the edges of the trihedral angle* $(c c_1 c_2)$; which latter angle is at once inscribed in the given cyclic cone, and also in that cone of revolution which it was originally proposed to construct.

29. Conceive now that this original cone of revolution $(c c_1 c_2)$ comes to *touch* the given cyclic cone along the side c, as a consequence of a gradual and unlimited approach of the second secant plane $(c c_2)$, to coincidence with the given tangent plane $(c' c d'')$, which touches the given cone along that side; or in virtue of a gradual and indefinite tendency of the side c_2 to coincide with the given side c. The line m, bisecting always the angle between these two sides c, c_2, will thus itself also tend to coincide with c; and the diagonal planes of the tetrahedral angle $(c'_1 d''_1 m' m'')$, which planes still intersect each other in m, will tend at the same time to contain the same given side. But that side c is (by the construction) a line in the plane of one face of that tetrahedral angle, namely in the plane of c'_1 and d''_1, which was the first secant plane of the cyclic cone; consequently the tetrahedral angle itself, and its circumscribed cone of revolution, tend generally to flatten together into coincidence with this secant plane, as c_2 thus approaches to c: and the axis of the cone $(c'_1 d''_1 m' m'')$ coincides ultimately with the normal to the first secant plane $(c'_1 d''_1)$. At the same time the traces c'_2 and d''_2, of the second secant plane on the two cyclic planes, tend to coincide with the traces c' and d'' of the given tangent plane thereupon. We have therefore this new theorem, which is however only a limiting form of that enunciated in article 27:—If through a given side (c) of a given cyclic cone, we draw a tangent plane $(c' c d'')$, and a secant plane $(c'_1 c c_1 d''_1)$; and if we then describe three cones of revolution, the first of these three cones containing on one sheet the two traces (c'_1, d''_1) of the secant plane, and one trace (d'') of the tangent plane; the second cone of revolution touching the cyclic cone along the side of contact (c), and cutting it along the side of the section (c_1); and the third cone of revolution containing the same two traces (c'_1, d''_1) of the secant plane, and the other trace (c') of the tangent plane: *the fourth harmonical to the axes of revolution of these three cones will be perpendicular to the secant plane.*

30. Finally, conceive that the remaining secant plane $c'_1 d''_1$ tends likewise to coincide with the tangent plane $c' d''$; the cone of revolution which lately *touched* the given cyclic cone along the given side c, will now come to *osculate* to that cone along that side: and because a line in the direction of the mutual intersection of the two cyclic planes has been already denoted by a', therefore the first and third of the three last-mentioned cones of revolution

tend now to touch the planes a′d″ and a′c′, respectively, along the lines d″ and c′. The theorem of article 27, at the limit here considered, takes therefore this new form:—*If three cones of revolution be described, the first cone cutting the first cyclic plane* (a′c′) *along the first trace* (c′) *of a given tangent plane* (c′cd″) *to a given cyclic cone, and touching the second cyclic plane* (a′d″) *along the second trace* (d″) *of the same tangent plane; the second cone of revolution osculating to the same cyclic cone, along the given side of contact* (c)*; and the third cone of revolution touching the first cyclic plane and cutting the second cyclic plane, along the same two traces as before: then the fourth harmonical to the axes of revolution of these three cones will be the normal to the plane* (c′d″) *which touches at once the given cyclic cone, and the sought osculating cone, along the side* (c) *of contact or of osculation.*

31. To deduce from this last theorem an *expression* for a line e in the direction of the axis of the osculating cone of revolution, by the processes of this symbolical geometry, we may remark in the first place, that when any two straight lines a, b, are equally long, we have the three equations following:

$$\mathrm{S}\frac{\mathrm{a}}{\mathrm{b}} = \mathrm{S}\frac{\mathrm{b}}{\mathrm{a}}, \quad \mathrm{V}\frac{\mathrm{a}}{\mathrm{b}} = -\mathrm{V}\frac{\mathrm{b}}{\mathrm{a}}, \quad \mathrm{I}\frac{\mathrm{a}}{\mathrm{b}} = -\mathrm{I}\frac{\mathrm{b}}{\mathrm{a}} \qquad (218),$$

from the two former of which it may be inferred that the relation

$$\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{a}}{\mathrm{b}} = -\frac{\mathrm{V}}{\mathrm{S}} \cdot \frac{\mathrm{b}}{\mathrm{a}} \qquad (219)$$

holds good, not only when the two lines a, b, are thus equal in length, but generally for any two lines: because if we multiply or divide either of them by any scalar coefficient, we only change thereby in one common (scalar) ratio both the scalar and vector parts of their quotient, and so do not affect that other quotient which is obtained by dividing the latter of these two parts by the former. We may also obtain the equation (219), as one which holds good for any two straight lines a, b, under the form

$$\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}}\mathrm{V}\frac{\mathrm{a}}{\mathrm{b}} + \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}}\mathrm{S}\frac{\mathrm{a}}{\mathrm{b}} = 0 \qquad (220),$$

by operating with the characteristic V on the identity,

$$\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}} \cdot \frac{\mathrm{a}}{\mathrm{b}} + \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}} \cdot \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{b}}{\mathrm{b}} = 1 \qquad (221);$$

while if we operate on the same identity (221) by the characteristic S, we obtain this other general formula, which likewise holds good for any two straight lines a, b, whether equal or unequal in length, and will be useful to us on future occasions,

$$\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}}\mathrm{S}\frac{\mathrm{a}}{\mathrm{b}} + \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}}\mathrm{V}\frac{\mathrm{a}}{\mathrm{b}} = 1 \qquad (222).$$

Again, if there be three equally long lines, a, b, c, then since the principle contained in the third equation (218) gives

$$\mathrm{I}\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}} = \mathrm{I}\frac{\mathrm{b}}{\mathrm{c}} - \mathrm{I}\frac{\mathrm{a}}{\mathrm{c}} = \mathrm{I}\frac{\mathrm{c}}{\mathrm{a}} - \mathrm{I}\frac{\mathrm{c}}{\mathrm{b}} \qquad (223),$$

which last expression is only multiplied by a scalar when the line c is multiplied thereby; while the index of a geometrical fraction is (among other properties) a line perpendicular to both the numerator and denominator of the fraction; we see that the symbol $\mathrm{I}\frac{\mathrm{c}}{\mathrm{a}} - \mathrm{I}\frac{\mathrm{c}}{\mathrm{b}}$ denotes generally a line perpendicular to both c and b − a, if only the two lines a and b have their own lengths equal to each other, without any restriction being thereby laid on the length of c: this symbol denotes therefore, under this single condition, a straight line contained in a plane perpendicular to c, and having equal inclinations to a and b. Thus, under the same condition, the symbol $\mathrm{I}\frac{\mathrm{c}}{\mathrm{a}} - \mathrm{I}\frac{\mathrm{c}}{\mathrm{b}}$ may represent the axis d of a cone of revolution, which contains upon one sheet the two equally long lines a and b, while the third line c is in or parallel to the *single* cyclic plane of this *monocyclic cone*, or the plane of its circular base, or of one of its circular sections; or coincides with or is parallel to some tangent to such circular base or section. If then we know any other line a′, contained in the plane which touches this monocyclic cone along the side a, we may substitute for c, in this symbol $\mathrm{I}\frac{\mathrm{c}}{\mathrm{a}} - \mathrm{I}\frac{\mathrm{c}}{\mathrm{b}}$, that part or component of this new line a′ which is perpendicular to the side of contact a; and therefore may write with this view,

$$\mathrm{c} = \mathrm{V}\frac{\mathrm{a}'}{\mathrm{a}} \,.\, \mathrm{a} = \mathrm{a}' - \mathrm{S}\frac{\mathrm{a}'}{\mathrm{a}} \,.\, \mathrm{a} \qquad (224),$$

which will give

$$\mathrm{d} = \mathrm{I}\frac{\mathrm{a}'}{\mathrm{a}} - \mathrm{I}\frac{\mathrm{a}'}{\mathrm{b}} + \mathrm{S}\frac{\mathrm{a}'}{\mathrm{a}}\mathrm{I}\frac{\mathrm{a}}{\mathrm{b}} \qquad (225),$$

as a general expression for a line d in the direction of the axis of a cone of revolution which is touched by the plane aa′ along the side of contact a, and contains on the same sheet the equally long side b. We may also remark that because the normal plane to a cone of revolution, drawn along any side of that cone, contains the axis of revolution, so that the plane containing the axis and the side is perpendicular to the tangent plane, we have a relation between the three directions of a, a′, d, which does not involve the direction of b, and may be expressed by any one of the three following formulæ:—

$$\angle(\mathrm{a}', \mathrm{a}, \mathrm{d}) = \frac{\pi}{2}, \quad \mathrm{d} \perp \mathrm{V}\frac{\mathrm{a}'}{\mathrm{a}} \,.\, \mathrm{a}, \quad \mathrm{S}\frac{\mathrm{a}'}{\mathrm{d}} = \mathrm{S}\frac{\mathrm{a}'}{\mathrm{a}}\mathrm{S}\frac{\mathrm{a}}{\mathrm{d}} \qquad (226);$$

in each of which it is allowed to reverse the direction of d, or to change d to −d. (Compare the formulæ (168), for the notation of dihedral angles.) It may indeed by easily proved, without the consideration of any cone, that any one of these three formulæ (226) involves the other two; but we see also, by the recent reasoning, that they may all be deduced when an expression of the form (225) for d is given; or when this line d can be expressed in terms of a, a′, and of another line b which is supposed to have the same length as a, by any symbol which differs only from the form (225) through the introduction of a scalar coefficient.

These things being premised, if we change a, b, d, in this form (225), to c′, d″, n′, we find

$$\mathrm{n}' = \mathrm{I}\frac{\mathrm{a}'}{\mathrm{c}'} - \mathrm{I}\frac{\mathrm{a}'}{\mathrm{d}''} + \mathrm{S}\frac{\mathrm{a}'}{\mathrm{c}'}\mathrm{I}\frac{\mathrm{c}'}{\mathrm{d}''} \qquad (227),$$

as an expression for a line n′ in the direction of the axis of revolution of the cone which touches the first cyclic plane a′c′ along the first trace c′ of the tangent plane, and cuts the second cyclic plane $a'd''$ along the second trace d'' of the same tangent plane; that is to say, in the direction of the axis of the third cone of revolution, described in the enunciation of the theorem of article 30. Again, if we change a, b, d, in the same general formula (225), to d'', c', $-n''$, and attend to the third equation (218), we find

$$n'' = I\frac{a'}{c'} - I\frac{a'}{d''} + S\frac{a'}{d''}I\frac{c'}{d''} \qquad (228),$$

as an expression for another line n'', in the direction of the axis of another cone of revolution, which cuts the first cyclic plane $a'c'$ along the trace c', and touches the second cyclic plane $a'd''$ along the other trace d'' of the tangent plane; that is, in the direction of the axis of revolution of the first of the three cones, described in the enunciation of the same theorem of article 30. And since these expressions give

$$n'' - n' = \left(S\frac{a'}{d''} - S\frac{a'}{c'}\right) I\frac{c'}{d''} \qquad (229),$$

we have the two perpendicularities

$$n'' - n' \perp c', \quad n'' - n' \perp d'' \qquad (230);$$

so that a transversal drawn across the two axes of revolution last determined, in the direction of this symbolic difference $n''-n'$, is perpendicular to both the traces of the tangent plane $c'd''$, and therefore has the direction of the normal to that plane, or to the cyclic cone; or, in other words, this transversal has the direction of the fourth harmonical mentioned in the theorem. But the lines n'' and n', of which the symbolic *difference* has thus been taken, have been seen to be in the directions of the first and third of the same four harmonicals; and the axis of the osculating cone, which axis we have denoted by e, has (by the theorem) the direction of the second harmonical: it has therefore the direction of the symbolical *sum* of the same two lines n'', n', because it bisects their transversal drawn as above. Thus by conceiving the bisector to terminate on the transversal, we find, as an expression for this sought axis e, the following,

$$e = \tfrac{1}{2}(n'' + n') = I\frac{a'}{c'} - I\frac{a'}{d''} + \tfrac{1}{2}\left(S\frac{a'}{c'} + S\frac{a'}{d''}\right) I\frac{c'}{d''} \qquad (231).$$

32. This symbolical expression for e contains, under a not very complex form, the solution of the problem on which we have been engaged; namely, *to find the axis of the cone of revolution, which osculates along a given side to a given cyclic cone.* It may however be a little simplified, and its general interpretation made easier, by resolving the line a' into two others, which shall be respectively parallel and perpendicular to the *lateral normal plane*, as follows:

$$a' = a^{\backprime} + a^{\backprime\backprime}; \quad a^{\backprime} \perp d'' - c'; \quad a^{\backprime\backprime} \parallel d'' - c' \qquad (232);$$

so that

$$a^{\backprime} = V\frac{a'}{d'' - c'} \,.\, (d'' - c'); \quad a^{\backprime\backprime} = S\frac{a'}{d'' - c'} \,.\, (d'' - c') \qquad (233);$$

which will give, by (191) and (218), because $a^{\backprime\backprime} \perp d'' + c'$,

$$\mathrm{S}\frac{d''}{a^{\backprime\backprime}} + \mathrm{S}\frac{c'}{a^{\backprime\backprime}} = 0; \quad \mathrm{S}\frac{a^{\backprime\backprime}}{c'} + \mathrm{S}\frac{a^{\backprime\backprime}}{d''} = 0 \qquad (234);$$

also

$$\mathrm{I}\frac{d''}{a^{\backprime\backprime}} - \mathrm{I}\frac{c'}{a^{\backprime\backprime}} = 0; \quad \mathrm{I}\frac{a^{\backprime\backprime}}{c'} - \mathrm{I}\frac{a^{\backprime\backprime}}{d''} = 0 \qquad (235);$$

and

$$\mathrm{S}\frac{d''}{a^{\backprime}} - \mathrm{S}\frac{c'}{a^{\backprime}} = 0; \quad \mathrm{S}\frac{a^{\backprime}}{c'} - \mathrm{S}\frac{a^{\backprime}}{d''} = 0 \qquad (236).$$

For by thus resolving a', in (231), into the two components $a^{\backprime}$ and $a^{\backprime\backprime}$, it is at once seen, by (234) (235), that the latter component $a^{\backprime\backprime}$ disappears from the result, which reduces itself by (236) to the following simplified form,

$$e = \mathrm{I}\frac{a^{\backprime}}{c'} - \mathrm{I}\frac{a^{\backprime}}{d''} + \mathrm{S}\frac{a^{\backprime}}{c'}\mathrm{I}\frac{c'}{d''} \qquad (237);$$

and this gives, by comparison with the forms (225) and (226), a remarkable relation of rectangularity between two planes, of which one contains the axis e of the osculating cone, namely the planes $a^{\backprime}c'$ and $c'e$; which relation is expressed by the formula,

$$\angle(a^{\backprime}, c', e) = \frac{\pi}{2} \qquad (238).$$

In like manner, from the same expression (231), by the same decomposition of a', we may easily deduce, instead of (237), this other expression for the axis of the osculating cone,

$$e = \mathrm{I}\frac{a^{\backprime}}{c'} - \mathrm{I}\frac{a^{\backprime}}{d''} - \mathrm{S}\frac{a^{\backprime}}{d''}\mathrm{I}\frac{d''}{c'} \qquad (239);$$

and may derive from it this other relation, of rectangularity between two other planes, namely the planes $a^{\backprime}d''$ and $d''e$,

$$\angle(a^{\backprime}, d'', e) = \frac{\pi}{2} \qquad (240).$$

Hence follows immediately this theorem, which furnishes a remarkably simple *construction with planes*, for determining generally a line in the required direction of the axis of the osculating cone:—*If we project the line* a' *of mutual intersection of the two cyclic planes* $a'c'$, $a'd''$*, of any given cyclic cone, on the lateral normal plane which is drawn along any given side* c*; if we next draw two planes,* $a^{\backprime}c'$, $a^{\backprime}d''$*, through the projection* $a^{\backprime}$ *thus obtained, and through the two traces,* c', d''*, of the tangent plane on the two cyclic planes; and if we then draw two new planes,* $c'e$, $d''e$*, through the same two traces of the tangent plane, perpendicular respectively to the two planes* $a^{\backprime}c'$, $a^{\backprime}d''$*, last drawn: these two new planes will intersect each other along the axis* e *of the cone of revolution, which osculates along the given side* c *to the given cyclic cone.*

And by considering, instead of these cones and planes, their intersection with a spheric surface described about the common vertex, we arrive at the following *spherographic construction*,* for finding the *spherical centre of curvature of a given spherical conic* at a given point, or the pole of the small circle which osculates at that point to that conic:—*From one of the two points of mutual intersection of the two cyclic arcs let fall a perpendicular upon the normal arc to the conic, which latter arc is drawn through the given point of osculation; connect the foot of this (arcual) perpendicular by two other arcs of great circles, with those two known points, equidistant from the point upon the conic, where the tangent arc meets the two cyclic arcs; draw through the same two points two new arcs of great circles, perpendicular respectively to the two connecting arcs: these two new arcs will cross each other on the normal arc, in the pole of the osculating circle, or in the spherical centre of curvature of the spherical conic*, which centre it was required to determine.

On Elliptic Cones, and on their Osculating Cones of Revolution.

33. With the same significations of a', $\mathrm{a}^{\backprime}$, c, c', d'', and e, as symbols of certain straight lines, connected with a given cyclic cone, as in the last article of this Essay; and with the same use of the sign I, as the characteristic of the *index* of the vector part of any geometrical fraction in general; if we now write

$$\mathrm{f} = \mathrm{I}\frac{\mathrm{a}'}{\mathrm{c}'}; \quad \mathrm{g} = \mathrm{I}\frac{\mathrm{d}''}{\mathrm{a}'}; \quad \mathrm{h} = \mathrm{I}\frac{\mathrm{d}''}{\mathrm{c}} = \mathrm{I}\frac{\mathrm{c}}{\mathrm{c}'} \qquad (241);$$

$$\mathrm{i} = \mathrm{I}\frac{\mathrm{a}^{\backprime}}{\mathrm{c}'}; \quad \mathrm{k} = \mathrm{I}\frac{\mathrm{d}''}{\mathrm{a}^{\backprime}}; \quad \mathrm{l} = \frac{\mathrm{h}}{\mathrm{c}}\mathrm{a}^{\backprime} \parallel \mathrm{I}\frac{\mathrm{a}'}{\mathrm{a}^{\backprime}} \qquad (242);$$

we shall thus form symbols for certain other straight lines, f, g, h, and i, k, l, which may be conceived to be all drawn from the same common origin as the former lines, namely from the vertex of the cyclic cone. And these new lines will be found to be connected with *another* cone, which may be called an *elliptic*† *cone*; namely the cone which is *normal*, *supplementary*, or

* This construction was communicated to the Royal Irish Academy (see *Proceedings*), at its meeting of November 30th, 1847, along with a simple geometrical construction for generating a system of two reciprocal ellipsoids by means of a moving sphere, as new applications of the author's Calculus of Quaternions to Surfaces of the Second Order. With that Calculus, of which the fundamental principles and formulæ were communicated to the same Academy on the 13th of November, 1843, it will be found that the present System of Symbolical Geometry is connected by very intimate relations, although the subject is approached, in the two methods, from two quite different points of view: the *algebraical quaternion* of the one method being *ultimately* the same as the *geometrical fraction* of the other.

† The methods of the present Symbolical Geometry might here be employed to prove that the *normal cone*, here called *elliptic*, from its connexion with its two focal lines, is itself *another cyclic cone*; being cut in circles by two sets of planes, which are perpendicular respectively to the two focal lines of the former cone. But it may be sufficient thus to have alluded to this well-known theorem, which it is not necessary for our present purpose to employ. There is even a convenience in retaining, for awhile, the two contrasted designations of *cyclic* and *elliptic*, for these two reciprocal cones, to mark more strongly the difference of the modes in which they here present themselves to our view.

reciprocal to the former *cyclic* cone. They may also be employed to assist in the determination of the *cone of revolution*, which *osculates* along a given side to this new or elliptic cone; as will be seen by the following investigation.

34. The lines f and g being, as is shewn by their expressions (241), perpendicular respectively to the planes a′c′ and a′d″, which were the two cyclic planes of the former or cyclic cone, are themselves the two *cyclic normals* of that cone; and because the line h is, by the same system of expressions (241), perpendicular to the plane c′d″ which touches that cyclic cone along the side c, it is the variable normal of that former cone: or this new line h is the *side* of the new or normal cone, which *corresponds* to that old side c. The inclinations of h to f and g, respectively, are given by the following equations, which are consequences of the same expressions (241):

$$\left.\begin{aligned}\angle(\mathrm{f},\mathrm{h}) &= \angle(\mathrm{a}',\mathrm{c}',\mathrm{c}) = \angle(\mathrm{a}',\mathrm{c}',\mathrm{d}'')\\ \angle(\mathrm{h},\mathrm{g}) &= \angle(\mathrm{a}',\mathrm{d}'',\mathrm{c}) = \angle(\mathrm{a}',\mathrm{d}'',\mathrm{c}')\end{aligned}\right\} \qquad (243);$$

and we have seen, in article 24, that for the cyclic cone an equation which may now be thus written holds good:

$$\angle(\mathrm{a}',\mathrm{c}',\mathrm{d}'') + \angle(\mathrm{a}',\mathrm{d}'',\mathrm{c}') = 2a \qquad (244);$$

where a is a constant angle: therefore for the cone of normals to that cyclic cone, the following other equation is satisfied:

$$\angle(\mathrm{f},\mathrm{h}) + \angle(\mathrm{h},\mathrm{g}) = 2a \qquad (245);$$

a being here the same constant as before. The sum of the inclinations of the variable side h of the new or *elliptic* cone to the two fixed lines f and g is therefore constant; in consequence of which known property, these two fixed lines are called the *focal lines* of the elliptic cone. And we see that these two *focal lines* f, g, of the *normal* cone, coincide, respectively, in their directions, with the two *cyclic normals* (or with the normals to the two cyclic planes) of the *original* cone: which is otherwise known to be true.

35. Another important and well-known property of the elliptic cone may be proved anew by observing that the expressions (241) give

$$\left.\begin{aligned}\angle(\mathrm{f},\mathrm{h},\mathrm{c}) &= \angle(\mathrm{f},\mathrm{h},\mathrm{c}') - \angle(\mathrm{c},\mathrm{h},\mathrm{c}') = \tfrac{1}{2}\pi - \angle(\mathrm{c},\mathrm{c}')\\ \angle(\mathrm{c},\mathrm{h},\mathrm{g}) &= \angle(\mathrm{d}'',\mathrm{h},\mathrm{g}) - \angle(\mathrm{d}'',\mathrm{h},\mathrm{c}) = \tfrac{1}{2}\pi - \angle(\mathrm{d}'',\mathrm{c})\end{aligned}\right\} \qquad (246);$$

and that we have, by (190),

$$\angle(\mathrm{d}'',\mathrm{c}) = \angle(\mathrm{c},\mathrm{c}') \qquad (247);$$

for thus we see that

$$\angle(\mathrm{f},\mathrm{h},\mathrm{c}) = \angle(\mathrm{c},\mathrm{h},\mathrm{g}) \qquad (248);$$

that is to say, the lateral normal plane hc to the reciprocal or elliptic cone (which is at the same time the lateral normal plane of the original or cyclic cone) bisects the dihedral angle

$\angle(\mathrm{f}, \mathrm{h}, \mathrm{g})$, comprised between the two *vector planes*, fh, hg, which connect the side h of the elliptic cone with the two focal lines f and g.

Or, because the expressions (241) shew that these two vector planes, fh, hg, of the elliptic cone, are perpendicular respectively to the two *traces* c' and d'' of the tangent plane to the cyclic cone, on the two cyclic planes of that cone; which traces are, as the formula (247) expresses, inclined equally to the side of contact c of the original or cyclic cone, while that side or line c is also the normal to the reciprocal or elliptic cone; we might hence infer that the tangent plane to the latter cone is equally inclined to the two vector planes: which is another form of the known relation.

36. The expressions (242), combined with (241), shew that the two new lines i and k, as being perpendicular respectively to the two traces c' and d'', are contained respectively in the two vector planes fh and hg. But each of the same two lines, i, k, is also perpendicular to the line $\mathrm{a}^{\backprime}$, to which the remaining new line l is also perpendicular, as the same expressions shew; they shew too that $\mathrm{a}^{\backprime}$ is a line in the common lateral and normal plane ch of the two cones, while l is also contained in that plane: the plane ik therefore cuts the plane ch perpendicularly in the line l. This latter line l is also, by the same expressions, perpendicular to the line a' (that is to the intersection of the two cyclic planes of the cyclic cone), which is perpendicular to both f and g; and therefore l can be determined, as the intersection of the common normal plane ch with the plane of the two focal lines fg; after which, by drawing through the line l, thus found, a plane ik perpendicular to ch, the lines i and k may be obtained, as the respective intersections of this last perpendicular plane with the two vector planes, fh, hg. And we see that these three new lines, i, k, l, introduced by the expressions (242), are such as to satisfy the following conditions of dihedral perpendicularity:

$$\tfrac{1}{2}\pi = \angle(\mathrm{h}, \mathrm{l}, \mathrm{i}) = \angle(\mathrm{k}, \mathrm{l}, \mathrm{h}) \qquad (249);$$

$$\tfrac{1}{2}\pi = \angle(\mathrm{h}, \mathrm{i}, \mathrm{c}') = \angle(\mathrm{d}'', \mathrm{k}, \mathrm{h}) \qquad (250);$$

$$\tfrac{1}{2}\pi = \angle(\mathrm{a}^{\backprime}, \mathrm{c}', \mathrm{i}) = \angle(\mathrm{a}^{\backprime}, \mathrm{d}'', \mathrm{k}) \qquad (251);$$

with which we may combine the following relations:

$$\angle(\mathrm{f}, \mathrm{h}, \mathrm{i}) = \angle(\mathrm{k}, \mathrm{h}, \mathrm{g}) = 0; \quad \angle(\mathrm{f}, \mathrm{l}, \mathrm{g}) = \pi;$$

$$\angle(\mathrm{f}, \mathrm{h}, \mathrm{l}) = \angle(\mathrm{l}, \mathrm{h}, \mathrm{g}) \qquad (252).$$

37. The positions of these three lines i, k, l, being thus fully known, by means of the expressions (242), or of the corollaries which have been deduced from those expressions, let us now consider, in connexion with them, the two formulæ of dihedral perpendicularity, (238), (240), which were given in article 32, to determine the axis e of a cone of revolution, which osculates along the side c to the given cyclic cone, and which formulæ may be thus collected:

$$\tfrac{1}{2}\pi = \angle(\mathrm{a}^{\backprime}, \mathrm{c}', \mathrm{e}) = \angle(\mathrm{a}^{\backprime}, \mathrm{d}'', \mathrm{e}) \qquad (253).$$

The comparison of (253) with (251) shews that the planes $\mathrm{c}'\mathrm{e}$, $\mathrm{d}''\mathrm{e}$, must coincide respectively with the planes $\mathrm{c}'\mathrm{i}$, $\mathrm{d}''\mathrm{k}$; because they are drawn like them respectively through the lines c', d″, and are like them perpendicular respectively to the planes a‵c′, a‵d″; the line e must therefore be the intersection of the two planes c′i, d″k, which contain respectively the two lines i, k, and are, by (250), perpendicular to the two planes ih, kh, or (by what has been seen in the last article) to the two vector planes fh, gh. We can therefore construct the line e as the intersection of the two planes ie, ke, which are thus drawn through the lately determined lines i, k, at right angles to the two vector planes; and we may write, instead of (253), the formulæ

$$\tfrac{1}{2}\pi = \angle(\mathrm{h}, \mathrm{i}, \mathrm{e}) = \angle(\mathrm{h}, \mathrm{k}, \mathrm{e}) \qquad (254).$$

38. Again, because this line e is (by Art. 32) the axis of a cone of revolution which *osculates* to the given cyclic cone, or which touches that cone not only along the side c itself but also along another side infinitely near thereto; while, in general, the lateral normal planes of a cone of revolution all cross each other along the axis of that cone; it is clear that e must be the line along which the common lateral and normal plane ch of the two reciprocal cones is intersected by an infinitely near normal and lateral plane to the first or cyclic cone, which is also at the same time a lateral and normal plane to the second or elliptic cone; consequently *the two cones of revolution which osculate to these two reciprocal cones, along these two corresponding sides*, c and h, *have one common axis*, e. And it is evident that a similar result for a similar reason holds good, in the more general case of *any two reciprocal cones*, which have a common vertex, and of which each contains upon its surface all the normals to the other cone, *however arbitrary the form* of either cone may be; any two such cones having always *one common system of lateral and normal planes*, and *one common conical envelope of all those normal planes*: which common envelope is thus the *common conical surface of centres of curvature*, for the two reciprocal cones.

Eliminating therefore what belongs, in the present question, to the original or cyclic cone, or confining ourselves to the formulæ (245), (249), (252), (254), we are conducted to the following *construction, for determining the axis* e *of that new cone of revolution, which osculates along a given side* h *to a given elliptic cone*; this latter cone having f and g for its given focal lines, or being represented by an equation of the form (245):—*Draw, through the given side,* h*, the normal plane* hl*, bisecting the angle between the two vector planes,* fh*,* gh*, and meeting in the line* l *the plane* fg *of the two given focal lines; through the same line* l *draw another plane* ik*, perpendicular to the normal plane* hl*, and cutting the vector planes in two new lines,* i *and* k*; through these new lines draw two new planes,* ie*,* ke*, perpendicular respectively to the two vector planes* fi*,* gk*, or* fh*,* gh*: these two planes will cross each other on the normal plane, in the sought axis* e *of the osculating cone of revolution.*

39. Or if we prefer to consider, instead of cones and planes, their intersections with a spheric surface described about the common vertex, as its centre; we then arrive at the following *spherographic construction, for finding the spherical centre of curvature of a given spherical ellipse*, at any given point of that curve, which may be regarded as being the *reciprocal* of the construction assigned at the end of the 32nd article of this essay:—Draw, from the given point H, of the ellipse, the normal arc HL, bisecting the spherical angle FHG between the two vector arcs FH, GH, and terminated at L by the arc FG which connects the two given foci, F and G; through L draw an arc of a great circle IK, perpendicular to the normal arc HL, and cutting one of the two vector arcs HF, HG, and the other of those two

vector arcs prolonged, in two new points, I and K; through these two new points draw two new arcs of great circles, IE, KE, perpendicular respectively to the two vector arcs, or to the arcs HI, HK: *the two new arcs so drawn will cross each other on the normal arc (prolonged), in a point* E, *which will be the spherical centre of curvature sought*, or the pole of the small circle which osculates at the given point H to the given spherical ellipse.

And since it is obvious (on account of the spherical right angles HIE, HKE, in the construction), that the points I, K are the respective middle points of those portions of the vector arcs, or of those arcs prolonged, which are comprised within this osculating circle; so that the arc IK, which has been seen to pass through the point L, and which crosses at that point L the arcual major axis of the ellipse (because that axis passes through both foci), is the *common bisector* of these two intercepted portions of the vector arcs, which intercepted arcs of great circles may be called (on the sphere) the two *focal chords of curvature* of the spherical ellipse; we are therefore permitted to enunciate the following *theorem*,* which is in general sufficient for the determination of the spherical centre of curvature, or pole of the osculating small circle, at any proposed point of any such ellipse:—*The great circle which bisects the two focal (and arcual) chords of curvature of any spherical ellipse, for any point of osculation, intersects the (arcual) axis major in the same point in which that axis is cut by the (arcual) normal to the ellipse, drawn at the point of osculation.*

On the Tensor of a Geometrical Quotient.

40. The equations (218) (222), of Art. 31, shew that for any two equally long straight lines, a, b, the following relation holds good,

$$\left(\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}}\right)^2 - \left(\mathrm{V}\frac{\mathrm{b}}{\mathrm{a}}\right)^2 = 1 \qquad (255);$$

or, more concisely, that

$$\mathrm{T}\frac{\mathrm{b}}{\mathrm{a}} = 1 \qquad (256),$$

if we introduce a new characteristic T of operation on a geometrical quotient, defined by the general formula,

$$\mathrm{T}\frac{\mathrm{b}}{\mathrm{a}} = \sqrt{\left\{\left(\mathrm{S}\frac{\mathrm{b}}{\mathrm{a}}\right)^2 - \left(\mathrm{V}\frac{\mathrm{b}}{\mathrm{a}}\right)^2\right\}} \qquad (257);$$

* This theorem was proposed by the present writer, in June 1846, at the Examination for Bishop Law's Mathematical Premium, in Trinity College, Dublin; and it was shewn by him in a series of Questions on that occasion, which have since been printed in the Dublin University Calendar for 1847, (see p. LXX), among the University Examination Papers for the preceding year, that this theorem, and several others connected therewith (for example, that the trigonometric tangent of the focal half chord of curvature is the harmonic mean between the tangents of the two focal vector arcs), might be deduced by *spherical trigonometry*, from the known constancy of the sum of the two vector arcs, or focal distances, for any one spherical ellipse. But in the method employed in the present essay, no use whatever has hitherto been made of any formula of spherical or even plane trigonometry, any more than of the doctrine of coordinates.

where it is to be observed, that the expression of which the square root is taken is essentially a positive scalar, because the square of *every* scalar is *positive*, while the square of *every* vector is on the contrary a *negative* scalar, by the principles of the 12th article. Hence, generally, for *any two* straight lines a, b, of which the lengths are denoted by $\overline{\mathrm{a}}$, $\overline{\mathrm{b}}$, we have the equation,

$$\mathrm{T}\frac{\mathrm{b}}{\mathrm{a}} = \overline{\mathrm{b}} \div \overline{\mathrm{a}} \qquad (258);$$

because the expression (257) is doubled, tripled, or multiplied by any positive number, when the line b is multiplied by the same number, whatever it be, while the line a remains unchanged. This geometrical signification of the expression $\mathrm{T}\frac{\mathrm{b}}{\mathrm{a}}$, may induce us to name that expression the TENSOR of the geometrical quotient $\frac{\mathrm{b}}{\mathrm{a}}$, on which the characteristic T has operated; because this *tensor* is a number which directs us how to *extend* (directly or inversely, that is, in what ratio to lengthen or shorten) the denominator line a, in order to render it *as long* as the numerator line b: and it appears to the writer, that there are other advantages in adopting this name "tensor", with the signification defined by the formula (257). Adopting it, then, we might at once be led to see, by (258), from considerations of compositions of ratios between the lengths of lines, that in any multiplication of geometrical quotients among themselves, "the tensor of the product is equal to the product of the tensors." But to establish this important principle otherwise, we may observe that by the equations (87), (88), (99), (100), of Arts. 11, 13, the vector part γ of the product $c + \gamma$ of any two geometrical quotients, represented by the binomial forms $b+\beta$, $a+\alpha$, is changed to its own opposite, $-\gamma$, while the scalar part c of the same product remains unchanged, when we change the signs of the vector parts β, α, of the two factors, without changing their scalar parts b, a, and also *invert*, at the same time, the *order* of those factors; in such a manner that either of the two following *conjugate equations* includes the other:

$$\left.\begin{aligned} c + \gamma &= (b + \beta)(a + \alpha) \\ c - \gamma &= (a - \alpha)(b - \beta) \end{aligned}\right\} \qquad (259);$$

and these two conjugate equations give, by multiplication,

$$c^2 - \gamma^2 = (b^2 - \beta^2)(a^2 - \alpha^2) \qquad (260),$$

because the product $(a + \alpha)(a - \alpha) = a^2 - \alpha^2$ is scalar, so that

$$(a^2 - \alpha^2)(b - \beta) = (b - \beta)(a^2 - \alpha^2).$$

This product, $a^2 - \alpha^2$, of the two *conjugate expressions*, or *conjugate geometrical quotients*, denoted here by

$$a + \alpha, \quad a - \alpha \qquad (261),$$

is not only scalar, but is also *positive*; because we have, by the principles of the 12th article, the two inequalities,

$$a^2 > 0, \quad \alpha^2 < 0 \qquad (262).$$

Making then, in conformity with (257),

$$\mathrm{T}(a+\alpha) = \mathrm{T}(a-\alpha) = \sqrt{(a^2-\alpha^2)} \tag{263},$$

we see that either of the two conjugate equations (259) gives, by (260),

$$\mathrm{T}(c+\gamma) = \mathrm{T}(b+\beta)\,.\,\mathrm{T}(a+\alpha) \tag{264};$$

or eliminating $c+\gamma$,

$$\mathrm{T}(b+\beta)(a+\alpha) = \mathrm{T}(b+\beta)\,.\,\mathrm{T}(a+\alpha) \tag{265}.$$

It is easy to extend this result to any number of geometrical quotients, considered as factors in a multiplication; and thus to conclude generally that, as already stated, *the tensor of the product is equal to the product of the tensors*; a theorem which may be concisely expressed by the formula,

$$\mathrm{T}\prod = \prod\mathrm{T} \tag{266}.$$

On Conjugate Geometrical Quotients.

41. It will be found convenient here to introduce a new characteristic, K, to denote the operation of passing from any geometrical quotent to its *conjugate*, by preserving the scalar part unchanged, but changing the sign of the vector part; with which new characteristic of operation K, we shall have, generally,

$$\mathrm{K}\frac{\mathrm{b}}{\mathrm{a}} = \mathrm{S}\frac{\mathrm{b}}{\mathrm{a}} - \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}} \tag{267};$$

or,

$$\mathrm{K}(a+\alpha) = a-\alpha \tag{268},$$

if a be still understood to denote a scalar, but α a vector quotient. The *tensors of two conjugate quotients are equal to each other*, by (263); so that we may write

$$\mathrm{TK}\frac{\mathrm{b}}{\mathrm{a}} = \mathrm{T}\frac{\mathrm{b}}{\mathrm{a}},\ \text{or briefly, } \mathrm{TK} = \mathrm{T} \tag{269};$$

and *the product of any two such conjugate quotients is equal to the square of their common tensor*,

$$\frac{\mathrm{b}}{\mathrm{a}}\mathrm{K}\frac{\mathrm{b}}{\mathrm{a}} = \left(\mathrm{T}\frac{\mathrm{b}}{\mathrm{a}}\right)^2 \tag{270}.$$

By separation of symbols, we may write, instead of (267),

$$\mathrm{K} = \mathrm{S} - \mathrm{V} \tag{271},$$

and the characteristic K is a *distributive symbol*, because S and V have been already seen to be such: so that the equations (74) (75), of Art. 10, give now the analogous equations,

$$\mathrm{K}\sum = \sum\mathrm{K}, \quad \mathrm{K}\Delta = \Delta\mathrm{K}, \tag{272},$$

or in words, *the conjugate of a sum* (of any number of geometrical quotients) *is the sum of the conjugates*; and in like manner, the conjugate of a difference is equal to the difference of the conjugates. But also we have seen, in (178), that

$$1 = \mathrm{S} + \mathrm{V},$$

because a geometrical quotient is always equal to the sum of its own scalar and vector parts; we may therefore now form the following *symbolical expressions for our two old characteristics of operation, in terms of the new characteristic* K,

$$\left.\begin{aligned} \mathrm{S} &= \tfrac{1}{2}(1+K) \\ \mathrm{V} &= \tfrac{1}{2}(1-K) \end{aligned}\right\} \qquad (273).$$

We may also observe that

$$\mathrm{KK}\frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{b}}{\mathrm{a}}, \text{ or } \mathrm{K}^2 = 1 \qquad (274);$$

the *conjugate of the conjugate* of any geometrical quotient being equal to that quotient itself. Combining (273), (274), we find, by an easy symbolical process, which the formulæ (272) shew to be a legitimate one,

$$\left.\begin{aligned} \mathrm{KS} &= \tfrac{1}{2}(\mathrm{K}+\mathrm{K}^2) = \tfrac{1}{2}(\mathrm{K}+1) = +\mathrm{S} \\ \mathrm{KV} &= \tfrac{1}{2}(\mathrm{K}-\mathrm{K}^2) = \tfrac{1}{2}(\mathrm{K}-1) = -\mathrm{V} \end{aligned}\right\} \qquad (275);$$

and accordingly the operation of *taking the conjugate* has been defined to consist in changing the sign of the vector part, without making any change in the scalar part, of the quotient on which the operation is performed. From (273), (274), we may also infer the symbolical equations,

$$\left.\begin{aligned} \mathrm{S}^2 &= \tfrac{1}{4}(1+\mathrm{K})^2 = \tfrac{1}{2}(1+\mathrm{K}) = \mathrm{S} \\ \mathrm{V}^2 &= \tfrac{1}{4}(1-\mathrm{K})^2 = \tfrac{1}{2}(1-\mathrm{K}) = \mathrm{V} \\ \mathrm{SV} = \mathrm{VS} &= \tfrac{1}{4}(1-\mathrm{K}^2) = 0 \end{aligned}\right\} \qquad (276);$$

and in fact, after once separating the scalar and vector parts of any proposed geometrical quotient, no farther separation of the same kind is possible; so that the operation denoted by the characteristic S, if it be again performed, makes no change in the scalar part first found, but reduces the vector part to zero; and, in like manner, the operation V reduces the scalar part to zero, while it leaves unchanged the vector part of the first or proposed quotient. We may note here that the same formulæ give these other symbolical results, which also can easily be verified:

$$\mathrm{KS} = \mathrm{SK}; \quad \mathrm{KV} = \mathrm{VK} \qquad (277);$$

and

$$(\mathrm{S}+\mathrm{V})^n = \mathrm{S}^n + \mathrm{V}^n = \mathrm{S} + \mathrm{V} = 1 \qquad (278);$$

at least if the exponent n be any positive whole number, so as to allow a finite and integral development of the symbolic power

$$(\mathrm{S}+\mathrm{V})^n = 1^n \qquad (279).$$

With respect to the *geometrical signification* of the relation between conjugate quotients, we may easily see that if c and d denote any two equally long straight lines, and x any scalar coefficient or multiplier, then the two quotients

$$\frac{x\mathrm{c}}{\mathrm{c}+\mathrm{d}}, \quad \frac{x\mathrm{d}}{\mathrm{c}+\mathrm{d}} \tag{280}$$

will be, in the foregoing sense, *conjugate*; because their *sum* will be a *scalar*, namely x, but their *difference* will be a *vector*, on account of the mutual perpendicularity of the lines $\mathrm{c}-\mathrm{d}$ and $\mathrm{c}+\mathrm{d}$, which are here the diagonals of a rhombus, and of which the latter bisects the angle between the sides c and d of the rhombus. (Compare (209).)

Conversely, if the relation

$$\frac{\mathrm{b}'}{\mathrm{a}} = \mathrm{K}\frac{\mathrm{b}}{\mathrm{a}} \qquad (281),$$

be given, we shall have, by the definition (267) of K,

$$0 = \mathrm{V}\frac{\mathrm{b}'+\mathrm{b}}{\mathrm{a}} = \mathrm{S}\frac{\mathrm{b}'-\mathrm{b}}{\mathrm{a}} \qquad (282);$$

whence it is easy to infer that, *if two conjugate geometrical quotients or fractions be so prepared as to have a common denominator* (a), *their numerators* (b, b′) *will be equally long, and will be equally inclined to the denominator, at opposite sides thereof, but in one common plane with it*; in such a manner that the line a (or −a) *bisects the angle* between the lines b and b′, if these three straight lines be supposed to have all one common origin. We are then conducted, in this way, to a very simple and useful *expression, for* (what may be called) *the reflexion* (b′) *of a straight line* (b), *with respect to another straight line* (a), namely the following:

$$\mathrm{b}' = \mathrm{K}\frac{\mathrm{b}}{\mathrm{a}} \times \mathrm{a} \qquad (283).$$

And whenever we meet with an expression of this form, we shall know that the two lines b and b′ are equally long; and also that if they have a common origin, the angle between them is bisected there by one of the two opposite lines ±a, or by a parallel thereto.

Finally, we may here note that, by the principles of the present article, and of the foregoing one, we have the following expressions, which hold good for any pair of straight lines, a and b:

$$\left.\begin{aligned} \frac{\mathrm{a}}{\mathrm{b}} &= \left(\mathrm{T}\frac{\mathrm{a}}{\mathrm{b}}\right)^2 \mathrm{K}\frac{\mathrm{b}}{\mathrm{a}} \\ \mathrm{S}\frac{\mathrm{a}}{\mathrm{b}} &= \left(\mathrm{T}\frac{\mathrm{a}}{\mathrm{b}}\right)^2 \mathrm{S}\frac{\mathrm{b}}{\mathrm{a}} \\ \mathrm{V}\frac{\mathrm{a}}{\mathrm{b}} &= -\left(\mathrm{T}\frac{\mathrm{a}}{\mathrm{b}}\right)^2 \mathrm{V}\frac{\mathrm{b}}{\mathrm{a}} \end{aligned}\right\} \qquad (284).$$

Equations of some Geometrical Loci.

42. The equation

$$\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} = 0 \qquad (285),$$

signifies, by what has been already shewn, that the straight line r is perpendicular to a; it is therefore the equation of a *plane*, perpendicular to this latter line, and passing through some fixed origin of lines, if r be regarded as a variable line, but a as a fixed line from that origin. The equation

$$\mathrm{S}\frac{\mathrm{r}-\mathrm{a}}{\mathrm{a}} = 0, \quad \textit{or} \quad \mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} = 1 \qquad (286),$$

expresses, for a similar reason, that the variable line r terminates on another plane, parallel to the former plane, and having the line a for the perpendicular let fall upon it from the origin. If b denote the perpendicular let fall from the same origin upon a third plane, the equation of this third plane will of course be, in like manner

$$\mathrm{S}\frac{\mathrm{r}}{\mathrm{b}} = 1 \qquad (287);$$

and it is not difficult to prove, with the help of the transformations (284), that this other equation

$$\mathrm{S}\frac{\mathrm{r}}{\mathrm{b}} = \mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} \qquad (288),$$

represents a fourth plane, which passes through the intersection of the second and third planes just now mentioned, namely, the planes (286), (287), and through the origin.

In general, the equation

$$\mathrm{S}\left(\frac{\mathrm{r}}{\mathrm{a}} + \frac{\mathrm{r}}{\mathrm{a}'} + \frac{\mathrm{r}}{\mathrm{a}''} + \&\mathrm{c.}\right) = a \qquad (289),$$

expresses that r terminates on a fixed plane, if it be drawn from a fixed origin, and if the lines a, a′, a″, &c., and the number *a* be given. It may also be noted here that the equation of the plane which perpendicularly bisects the straight line connecting the extremities of two given lines, a and b, may be thus written:

$$\mathrm{T}\frac{\mathrm{r}-\mathrm{b}}{\mathrm{r}-\mathrm{a}} = 1 \qquad (290).$$

43. On the other hand, the equation

$$\mathrm{S}\frac{\mathrm{r}-\mathrm{b}}{\mathrm{r}-\mathrm{a}} = 0 \qquad (291),$$

expresses that the lines from the extremities of a and b to the extremity of r are perpendicular to each other; or that the line r terminates upon a *spheric surface*, in two diametrically opposite points of which surface the lines a and b respectively terminate: and this diameter itself, from the end of a to the end of b, regarded as a *rectilinear locus*, is represented by the equation

$$\mathrm{V}\frac{\mathrm{r}-\mathrm{b}}{\mathrm{r}-\mathrm{a}} = 0 \qquad (292);$$

which may however be put under other forms. A transformation of the equation (291) is the following:

$$\mathrm{T}\left(\frac{2\mathrm{r}-\mathrm{b}-\mathrm{a}}{\mathrm{b}-\mathrm{a}}\right) = 1 \qquad (293);$$

which expresses that the variable radius $\mathrm{r} - \frac{1}{2}(\mathrm{b} + \mathrm{a})$ has the same length as the fixed radius $\frac{1}{2}(\mathrm{b} - \mathrm{a})$. For example, by changing $-\mathrm{a}$ to $+\mathrm{b}$, in this last equation of the sphere, we find

$$\mathrm{T}\frac{\mathrm{r}}{\mathrm{b}} = 1, \quad \text{or} \quad \left(\mathrm{S}\frac{\mathrm{r}}{\mathrm{b}}\right)^2 - \left(\mathrm{V}\frac{\mathrm{r}}{\mathrm{b}}\right)^2 = 1 \qquad (294),$$

as the equation of a spheric surface described about the origin of lines, as centre, with the line b for one of its radii, so as to touch, at the end of this line b, the plane (287). (Comp. (255)).

And a small *circle* of this sphere (294), if it be situated on a secant plane, parallel to this tangent plane (287), which new plane will thus have for its equation,

$$\mathrm{S}\frac{\mathrm{r}}{\mathrm{b}} = x \qquad (295),$$

where x is a scalar, numerically less than unity, and constant for each particular circle, will also be situated on a certain corresponding cylinder of revolution, which will have for its equation

$$\left(\mathrm{V}\frac{\mathrm{r}}{\mathrm{b}}\right)^2 = x^2 - 1 \qquad (296);$$

where $x^2 - 1$ is negative, as it ought to be, by the 12^{th} article, being equal to the square of a vector. The sphere may be regarded as the locus of these small circles; and its equation (294) may be supposed to be obtained by the elimination of the scalar x between the equations of the plane (295), and of the cylinder (296).

44. Conceive now that instead of cutting the cylinder (296) *perpendicularly* in a *circle*, we cut it *obliquely*, in an *ellipse*, by the plane having for its equation

$$\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} = x \qquad (297),$$

where x is the same scalar as before; so that this new plane is parallel to the fixed plane (286), and cuts the plane of the circle (295) in a straight line situated on that other fixed plane (288), which has been seen to contain also the intersection of the same fixed plane (286) with the tangent plane (287). The *locus of the elliptic sections*, obtained from the circular cylinders by this construction, will be an *ellipsoid*; and conversely, an ellipsoid may in general be regarded as such a locus. The equation of the *ellipsoid*, thus found, by eliminating x between the equations (296), (297), is the following:

$$\left(\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}}\right)^2 - \left(\mathrm{V}\frac{\mathrm{r}}{\mathrm{b}}\right)^2 = 1 \qquad (298);$$

and by some easy modifications of the process, it may be shewn that a *hyperboloid*, regarded as a certain other locus of ellipses, may in general be represented by an equation of the form

$$\left(\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}}\right)^2 + \left(\mathrm{V}\frac{\mathrm{r}}{\mathrm{b}}\right)^2 = \mp 1 \qquad (299).$$

The upper sign belongs to a hyperboloid of *one sheet*, but the lower sign to a hyperboloid of *two sheets*; while the *common asymptotic cone* of these two (conjugate) hyperboloids (299) is the locus of a certain other system of ellipses, and is represented by the analogous but intermediate equation,

$$\left(\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}}\right)^2 + \left(\mathrm{V}\frac{\mathrm{r}}{\mathrm{b}}\right)^2 = 0 \qquad (300).$$

These equations admit of several instructive transformations, to some of which we shall proceed in the following article.

45. The equation (298) of the ellipsoid resolves itself into factors, as follows:

$$\left(\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} + \mathrm{V}\frac{\mathrm{r}}{\mathrm{b}}\right)\left(\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} - \mathrm{V}\frac{\mathrm{r}}{\mathrm{b}}\right) = 1 \qquad (301);$$

where the sum and difference, which when thus multiplied together give unity for their product, are *conjugate expressions* (in the sense of recent articles); they have therefore a *common tensor*, which must itself be equal to unity; and consequently we may write the equation of the ellipsoid thus,

$$\mathrm{T}\left(\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} + \mathrm{V}\frac{\mathrm{r}}{\mathrm{b}}\right) = 1 \qquad (302),$$

where the sign of the vector may be changed. Substituting for the characteristics of operation S and V, their symbolical values (273), we are led to introduce two new fixed lines g and h, depending on the two former fixed lines a and b, and determined by the equations

$$\frac{\mathrm{r}}{2\mathrm{a}} + \frac{\mathrm{r}}{2\mathrm{b}} = \frac{\mathrm{r}}{\mathrm{g}}; \quad \frac{\mathrm{r}}{2\mathrm{a}} - \frac{\mathrm{r}}{2\mathrm{b}} = \frac{\mathrm{r}}{\mathrm{h}} \qquad (303);$$

and thus the equation of the ellipsoid may be changed from (302) to this other form

$$\mathrm{T}\left(\frac{\mathrm{r}}{\mathrm{g}} + \mathrm{K}\frac{\mathrm{r}}{\mathrm{h}}\right) = 1 \qquad (304);$$

which, by the principles (269), (272), (274), may also be thus written,

$$\mathrm{T}\left(\frac{\mathrm{r}}{\mathrm{h}} + \mathrm{K}\frac{\mathrm{r}}{\mathrm{g}}\right) = 1 \qquad (305);$$

so that the symbols, g and h, may be interchanged in either of the two last forms of the equation of the ellipsoid.

46. Let $\bar{\mathrm{r}}$, $\bar{\mathrm{g}}$, $\bar{\mathrm{h}}$ be conceived to be numerical symbols, denoting respectively the lengths of the three lines r, g, h; and make, for conciseness,

$$\mathrm{r} \div \bar{\mathrm{r}}^2 = \mathrm{r}'; \quad \mathrm{g} \div \bar{\mathrm{g}}^2 = \mathrm{g}'; \quad \mathrm{h} \div \bar{\mathrm{h}}^2 = \mathrm{h}' \qquad (306);$$

so that the symbols r′, g′, h′ shall denote three new lines, having the *same directions* as the three former lines r, g, h, but having their *lengths* respectively *reciprocals* of the lengths of those three former lines. Then, by the properties of conjugate quotients already established, we shall have the transformations

$$\frac{\mathrm{r}}{\mathrm{g}} = \mathrm{K}\frac{\mathrm{g}'}{\mathrm{r}'}; \quad \mathrm{K}\frac{\mathrm{r}}{\mathrm{h}} = \frac{\mathrm{h}'}{\mathrm{r}'} \qquad (307);$$

whereby the equation (304) of the ellipsoid becomes

$$\mathrm{T}\left(\frac{\mathrm{h}'}{\mathrm{r}'} + \mathrm{K}\frac{\mathrm{g}'}{\mathrm{r}'}\right) = 1 \qquad (308).$$

Let g'' be a new line, not fixed but variable, and determined for each variable direction of r' or of r by the formula

$$\mathrm{g}'' = \mathrm{K}\frac{\mathrm{g}'}{\mathrm{r}'} \times \mathrm{r}'; \text{ or } \mathrm{g}'' = \mathrm{K}\frac{\mathrm{g}'}{\mathrm{r}} \times \mathrm{r} \qquad (309);$$

so that this new and variable line g'' is, by what was shewn respecting the expression (283), the *reflexion* of the fixed line g' with respect to a line having the variable direction just mentioned, of r' or of r: we may then write the equation (308) of the ellipsoid as follows,

$$\mathrm{T}\frac{\mathrm{h}' + \mathrm{g}''}{\mathrm{r}'} = 1 \qquad (310).$$

And by comparing this with the formula (256), we see that the length of the line r', or *the reciprocal of the length* $\bar{\mathrm{r}}$ *of the variable semidiameter* r *of the ellipsoid, is equal to the length of the line* $\mathrm{h}' + \mathrm{g}''$; which latter line is the symbolical sum of one fixed line, h', and of the variable reflexion, g'', of another fixed line, g'; this *reflexion* having been already seen to be performed with respect to the variable radius vector or semidiameter, r, of the ellipsoid, *of which semidiameter the dependence of the length on the direction admits of being thus represented, or constructed, by a very simple geometrical rule.*

47. To make more clear the conception of this geometrical rule, let A denote the centre of the ellipsoid, which centre is the origin of the variable line r; and let two other fixed points, B and C, be determined by the symbolical equations

$$\mathrm{g}' = \mathrm{A} - \mathrm{C} = \mathrm{AC}; \quad \mathrm{h}' = \mathrm{B} - \mathrm{C} = \mathrm{BC} \qquad (311):$$

these two notations, AC and A − C, (of which one has been already used in the *text* of the first article* of this Essay on Symbolical Geometry, while the other was suggested in a *note* to the same early article,) being each designed to denote or signify a straight line drawn *to* the point A from the point C. Let D be a new or fourth point, not fixed but variable, and determined by the analogous equation

$$\mathrm{g}'' = \mathrm{C} - \mathrm{D} = \mathrm{CD} \qquad (312):$$

* It was for the sake of making easier the transition to the notation B − A, which appears to the present writer an expressive one, for the straight line drawn *to* the point B *from* the point A, that he proposed to use, with the *same* geometrical signification, the symbol BA, instead of AB: although it is certainly more usual, and perhaps also more natural, when *direction* is attended to, to employ the latter symbol AB, and not the former BA, to denote the line thus drawn from A to B.

then because, in virtue of the relation (309), the lines g', g'' are equally long, it follows that the variable point D is situated on the surface of that fixed and *diacentric sphere*, which we may conceive to be described *round* the fixed point C as centre, so as to pass *through* the centre A of the ellipsoid as through a given superficial point of this diacentric sphere. Again, in virtue of the same relation (309), or of the geometric reflexion which the second formula so marked expresses, the symbolic sum of the two lines g', g'', has the direction of the line r, or the exactly contrary direction; in fact, that relation (309) conducts to the following scalar quotient

$$\frac{g' + g''}{r} = \frac{g'}{r} + K\frac{g'}{r} = 2S\frac{g'}{r} \qquad (313);$$

and this symbolic sum, $g' + g''$, may also, by (311) (312), be thus expressed

$$g' + g'' = (A - C) + (C - D) = A - D = AD \qquad (314).$$

If then we denote by E that variable point on the surface of the ellipsoid at which the line r terminates, so that

$$r = E - A = EA \qquad (315),$$

we shall have the relation

$$\frac{A - D}{E - A} = \frac{AD}{EA} = 2S\frac{g'}{r} = V^{-1}0 \qquad (316),$$

which requires that the three points, A, D, E, should be situated on one common straight line. We know then the geometrical position of the auxiliary and variable point D, or have a simple construction for determining this variable point D, as corresponding to any particular point E on the surface of the ellipsoid, when the centre A, and the two other fixed points, B and C, are given; for we see that we have merely to seek the *second intersection* of the semidiameter $E - A$ (or EA) of the ellipsoid, with the surface of the diacentric sphere, the *first* intersection being the centre A itself; since this second point of intersection will be the required point D.

48. But also, by (311) (312), we have

$$h' + g'' = (B - C) + (C - D) = B - D = BD \qquad (317):$$

this line BD has therefore, by (310), the length of the line r'; which length is, by (306), (315) the reciprocal of the length $\bar{r}$ of the semidiameter EA of the ellipsoid. The lines g, h have generally unequal lengths; and because, by (304) (305), their symbols may be interchanged, we may choose them so that the former shall be the longer of the two, or that the inequality

$$\bar{g} > \bar{h} \qquad (318)$$

shall be satisfied; and then, by (306), the line g' will, on the contrary, be shorter than the line h', or the fixed point B will be *exterior* to the fixed diacentric sphere. Drawing, then, from this external point B, a tangent to this diacentric sphere, and taking the length of the tangent so drawn for the unit of length, the reciprocal of the length of the line BD, which is considered in (317), will be the length of that other line BD′, which has the same direction as BD, but terminates at another variable point D′ on the surface of the diacentric sphere; in such a manner that this new variable point D′, without generally coinciding with the point D, shall satisfy the two equations,

$$\frac{\text{D}' - \text{B}}{\text{D} - \text{B}} = \text{V}^{-1}0; \quad \frac{\text{D}' - \text{C}}{\text{D} - \text{C}} = \text{T}^{-1}1 \qquad (319);$$

for then the two lines D′ − B, D − B (or D′B, DB) will be, in this or in the opposite order, the whole secant and external part, while the length of the tangent to the sphere has been above assumed as unity. Under these conditions, then, *the lengths of the lines* D′B *and* EA *will be equal*, because they will have the length of the line DB for their common reciprocal; so that we shall have the equation

$$\frac{\text{E} - \text{A}}{\text{D}' - \text{B}} = \text{T}^{-1}1 \qquad (320);$$

or, in a more familiar notation,

$$\overline{\text{AE}} = \overline{\text{BD}'} \qquad (321).$$

It may be noted here that the new radius D′ − C of the diacentric sphere admits (compare the formula (213)) of being symbolically expressed as follows,

$$\text{D}' - \text{C} = \text{K}\frac{\text{g}''}{\text{h}' + \text{g}''} \,.\, (\text{h}' + \text{g}'') \qquad (322);$$

and, accordingly, this last expression satisfies the two conditions (319), because it gives

$$\frac{\text{D}' - \text{B}}{\text{D} - \text{B}} = \text{S}\frac{\text{h}' - \text{g}''}{\text{h}' + \text{g}''} \qquad (323),$$

and

$$\frac{\text{D}' - \text{C}}{\text{D} - \text{C}} = \text{K}\frac{\text{g}''}{\text{h}' + \text{g}''} \,.\, \frac{\text{h}' + \text{g}''}{-\text{g}''} \qquad (324),$$

of which latter expression the tensor is unity.

49. The remarkably simple formula,

$$\overline{\text{AE}} = \overline{\text{BD}'} \qquad (321),$$

to which we have thus been conducted for the ellipsoid, admits of being easily translated into the following rule for constructing that important surface; which rule for the *construction of the ellipsoid* does not seem to have been known to mathematicians, until it was communicated by the present writer to the Royal Irish Academy in 1846, as a result of his Calculus of Quaternions, between which and the present Symbolical Geometry a very close affinity exists.

From a fixed point A*, on the surface of a given sphere, draw a variable chord of that sphere,* DA*; let* D′ *be the second point of intersection of the spheric surface with the secant* DB*, which connects the variable extremity* D *of this chord* DA *with a fixed external point* B*; and take the radius vector* EA *equal in length to* D′B*, and in direction either coincident with, or* *opposite to, the chord* DA*: the locus of the point* E*, thus constructed, will be an ellipsoid, which will have its centre at the fixed point* A*, and will pass through the fixed point* B.

The fixed sphere through A, in this construction of the ellipsoid, is the *diacentric sphere* of recent articles; it may also be called a *guide-sphere*, from the manner in which it assists to mark or to *represent the direction*, and at the same time serves to *construct the length* of a variable semidiameter of the ellipsoid; while, for a similar reason, the points D and D′ upon the surface of this sphere may be said to be *conjugate guide-points*; and the chords DA and D′A may receive the appellation of *conjugate guide-chords.* In fact, while either of these two guide-chords of the sphere, for instance (as above) the chord DA, coincides in *direction* with a semidiameter EA of the ellipsoid, the distance $\overline{\text{D}'\text{B}}$ of the extremity D′ of the other or conjugate guide-chord, D′A, from the fixed external point B, represents, as we have seen, the *length* of that semidiameter. And that the fixed point B, although exterior to the diacentric sphere, is a superficial point of the ellipsoid, appears from the construction, by conceiving the conjugate guide-point D′ to approach to coincidence with A; for E will then tend to coincide either with the point B itself, or with another point diametrically opposite thereto, upon the surface of the ellipsoid.

50. Some persons may prefer the following mode of stating the same geometrical construction, or the same fundamental property, of the ellipsoid: which other mode also was communicated by the present writer to the Royal Irish Academy in 1846. *If, of a rectilinear quadrilateral* ABED′*, of which one side* AB *is given in length and in position, the two diagonals* AE*,* BD′ *be equal to each other in length, and intersect* (in D) *on the surface of a given sphere* (with centre C), *of which sphere a chord* AD′ *is a side of the quadrilateral adjacent to the given side* AB*, then the other side* BE*, adjacent to the same given side* AB*, is a chord of a given ellipsoid.*

Thus, denoting still the centre of the sphere by C, while A is still the centre of the ellipsoid, we see that the form, magnitude, and position, of this latter surface are made by the foregoing construction to depend, according to very simple geometrical rules, on the positions of the three points A, B, C; or on the form, magnitude, and position of what may (for this reason) be named the *generating triangle* ABC. Two of the sides of this triangle, namely BC and CA, are perpendicular, as it is not difficult to shew from the construction, to the two *planes of circular section* of the ellipsoid; and the third side AB is perpendicular to one of the two *planes of circular projection* of the same ellipsoid: this third side AB being the axis of revolution of a circumscribed circular cylinder; which also may be proved, without difficulty, from the construction assigned above. (See Articles 52, 53.) The length $\overline{\text{BC}}$ of the side BC of the triangle, is (by the construction) the semisum of the lengths of the greatest and least semidiameters of the ellipsoid; and the length $\overline{\text{CA}}$ of the side CA is the semidifference of the lengths of those extreme semidiameters, or principal semiaxes, of the same ellipsoid: while (by the same construction) these greatest and least *semiaxes*, or their prolongations, intersect the surface of the diacentric sphere in points which are situated, respectively, on the finite side CB of the triangle ABC itself, and on that side CB prolonged through C. The *mean* semiaxis of the ellipsoid, or the semidiameter perpendicular to the greatest and least semiaxes, is (by the construction) equal in length (as indeed it is otherwise known to be) to the radius of the enveloping cylinder of revolution, or to the radius of either of the two diametral and circular sections: the length of this mean semiaxis is also constructed by the portion $\overline{\text{BG}}$ of the axis of the enveloping cylinder, or of the side BA of the generating triangle, if G be the point, distinct from A, in which this side BA meets the surface of the diacentric sphere. And hence we may derive a simple geometrical signification, or property, of this remaining side BA of the triangle ABC, as respects its length $\overline{\text{BA}}$; namely, that this length is a fourth proportional to the three semiaxes of the ellipsoid, that is to say, to the mean, the least, and the greatest, or to the mean, the greatest, and the least of those three principal and rectangular semiaxes.

On tbe Law of the Variation of the Difference of the Squares of the Reciprocals of the Semiaxes of a Diametral Section.

51. To give a specimen of the facility with which the foregoing construction serves to establish some important properties of the ellipsoid, we shall here employ it to investigate anew the known and important law, according to which the difference of the squares of the reciprocals of the greatest and least semidiameters, of any plane and diametral section, varies in passing from one such section to another. Conceive then that the ellipsoid itself, and the auxiliary or diacentric sphere which was employed in the foregoing construction, are both cut by a plane $\text{AB}'\text{C}'$, passing through the centre A of the ellipsoid, and having B' and C' for the orthogonal projections, upon this secant plane, of the fixed points B and C. The auxiliary or guide-point D comes thus to be regarded as moving on the circumference of a circle, which passes through A, and has its centre at C': and since the semidiameter EA of the ellipsoid, as being equal in length to $\text{D}'\text{B}$, by the formula (321) of Art. 48, (or because these are the two equally long diagonals of the quadrilateral ABED' of Art. 50), must vary inversely as DB (by an elementary property of the sphere), we are led to seek the difference of the squares of the greatest and least values of DB, or of DB', since the square of the perpendicular $\text{B}'\text{B}$ is constant for the section. But the shortest and longest straight lines, $\text{D}_1\text{B}'$, $\text{D}_2\text{B}'$, which can be thus drawn to the circumference of the auxiliary circle round C' (namely the section of the diacentric sphere), from the fixed point B' in its plane, are those drawn to the extremities D_1, D_2 of that diameter $\text{D}_1\text{C}'\text{D}_2$ which passes through, or tends towards this point B'; in such a manner that the four points $\text{B}'\text{D}_1\text{C}'\text{D}_2$ are situated on one straight line. Hence the difference of the squares of $\text{D}_1\text{B}'$, $\text{D}_2\text{B}'$, is equal to four times the rectangle under $\text{D}_1\text{C}'$, or AC', and $\text{B}'\text{C}'$; that is to say, under the projections of the sides AC, BC, of the generating triangle, on the plane of the diametral section. *It is, then, to this rectangle, under these two projections of two fixed lines, on any variable plane through the centre of the ellipsoid, that the difference of the squares of the reciprocals of the extreme semidiameters of the section is proportional.* Hence, in the language of trigonometry, this difference of squares is proportional (as indeed it is well known to be) to the product of the sines of the inclinations of the cutting plane to two fixed planes of circular section; which latter planes are at the same time seen to be perpendicular to the two fixed sides AC, BC, of the generating triangle in the construction.

It seems worth noting here, that the foregoing process proves at the same time this other well-known property of the ellipsoid, that the greatest and least semidiameters of a plane section through the centre are perpendicular to each other; and also gives an easy geometrical rule for *constructing the semiaxes of any proposed diametral section*; for it shews that these semiaxes have the *directions* of the two rectangular guide-chords D_1A, D_2A; while their *lengths* are equal, respectively, to those of the lines $\text{D}'_1\text{B}$, $\text{D}'_2\text{B}$.

52. It may not be uninstructive to state briefly here some simple geometrical reasonings, by which the line BG of Art. 50 may be shewn to have its length equal to that of the radius of an enveloping cylinder of revolution, as was asserted in that article; and also to the radius of either of the two diametral and circular sections of the ellipsoid. First, then, as to the cylinder: the equation $\overline{\text{AE}} = \overline{\text{BD}'}$ shews that the rectangle under the two lines AE and BD is constant for the ellipsoid, because the rectangle under BD′ and BD is constant for the sphere; and the point D has been seen to be situated on the straight line AE (prolonged if necessary). Hence the double area of the triangle ABE, or the rectangle under the fixed line AB, and the perpendicular let fall thereon from the variable point E of the ellipsoid, is always less than the lately mentioned constant rectangle; or than the square of the tangent to the diacentric sphere from B; or, finally, than the rectangle under the same fixed line AB and its constant part GB: except at the limit where the angle ADB is right, at which limit the double area of the triangle ABE becomes equal to the last mentioned rectangle. The ellipsoid is therefore entirely enveloped by that cylinder of revolution which has AB for axis, and $\overline{\text{GB}}$ for radius; being situated entirely *within* this cylinder, except for a certain limiting curve or system of points, which are *on* (but not outside) the cylinder, and are determined by the condition that ADB shall be a right angle. This limiting condition determines a *second spherical locus* for the guide-point D, besides the diacentric sphere; it serves therefore to assign a *circular locus* for that point, which circle passes through the centre A of the ellipsoid, because this centre is situated on each of the two spherical loci. And hence by the construction we obtain an *elliptic locus* for the point E, namely the ellipse of contact of the ellipsoid and cylinder; which ellipse presents itself here as the intersection of that enveloping cylinder of revolution with the plane of the circle which has been seen to be the locus of D.—It may also be shewn, geometrically, by pursuing the same construction into its consequences, that the ellipsoid is enveloped by *another* (equal) cylinder of revolution, giving a *second* diametral *plane of circular projection*; the first such plane being (by what precedes) perpendicular to the line AB: and that the axis of this second circular cylinder, or the normal to this second plane of circular projection of the ellipsoid, is parallel to the straight line which touches, at the centre C of the diacentric sphere, the circle circumscribed about the generating triangle ABC.

53. Again, with respect to the diametral and circular sections of the ellipsoid, considered as results of the construction: if we conceive that the guide-point D, in that construction, approaches in any direction, on the surface of the diacentric sphere, to the centre A of the ellipsoid, the conjugate guide-point D′ must then approach to the point G, because this is the second point of intersection of the side BA of the triangle with the surface of the diacentric sphere, if the point A itself be regarded as the first point of such intersection. Thus, during this approach of D to A, the semidiameter EA of the ellipsoid, having always (by the construction) the direction of ±DA, and the length of D′B, must tend to touch the diacentric sphere at A, and to have the same fixed length as the line BG, or as the radius of the cylinder. And in this way the construction offers to our notice a *circle* on the ellipsoid, whose radius $= \overline{\text{BG}}$, and whose plane is perpendicular to the side AC of the generating triangle; which side is thus seen to be a *cyclic normal* of the ellipsoid, by this process as well as by that of the 51^{st} article.

Finally, with respect to that *other* cyclic plane which is perpendicular to the side BC of the triangle ABC, it is sufficient to observe that if we conceive the point D′ to revolve in a small circle on the surface of the diacentric sphere, from G to G again, preserving a constant distance from the fixed external point B, then the semidiameter EA of the ellipsoid will retain, by the construction, during this revolution of D′, a constant length = $\overline{\text{BG}}$; while, by the same construction, the guide-chord DA, and the semidiameter EA of the ellipsoid, will at the same time revolve together in a diametral plane perpendicular to BC: in which *second cyclic plane*, therefore, the point E will thus trace out a *second circle* on the ellipsoid, with a radius equal to the radius of the former circle; or to that of the *mean sphere* (constructed on the mean axis as diameter, and containing both the circles hitherto considered); or to the radius of either of the two enveloping cylinders of revolution.—It is evident that if the guide-point D describe any other circle on the diacentric sphere, parallel to this second cyclic plane, the conjugate guide-point D′ will describe another parallel circle, leaving the length $\overline{\text{BD}'} = \overline{\text{EA}}$ unaltered; whence the known theorem flows at once, that if the ellipsoid be cut by a concentric sphere, the section is a spherical ellipse;* and also that the concentric cyclic cone which rests thereon (being the cone described by the guide-chord DA in the construction) has its two cyclic planes coincident with the two cyclic planes of the ellipsoid.

* This easy mode of deducing, from the author's construction of the ellipsoid, the known spherical ellipses on that surface, was pointed out to him in 1846, by a friend to whom he had communicated that construction, namely by the Rev. J. W. Stubbs, Fellow of Trinity College, Dublin. Several investigations, by the present author, connected with the same construction of the ellipsoid, have appeared in the *Proceedings* of the Royal Irish Academy, (see in particular those for July 1846); and also in various numbers of the (London, Edinburgh and Dublin) *Philosophical Magazine*: in which magazine several articles on Quaternions have been already published by the writer, and are likely to be hereafter continued, which may on some points be usefully compared with the present Essay on Symbolical Geometry.