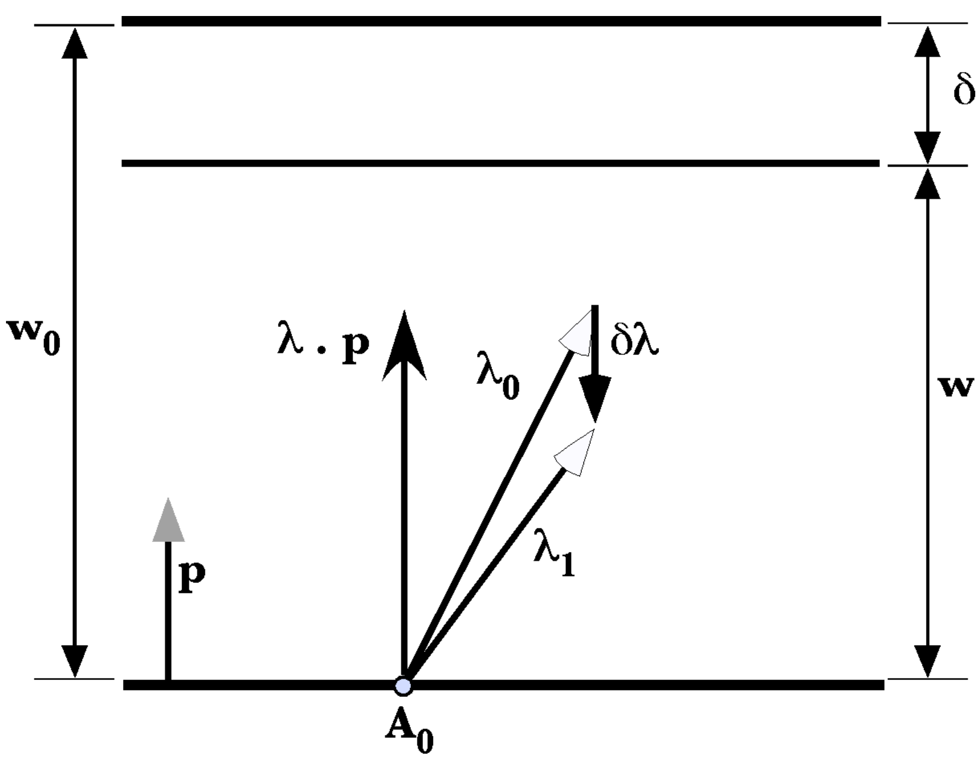
Compression Between Parallel Plates

Consider two flat plates that enclose an elastic amorphous material that can flow, but is nearly incompressible and inextensible. The two plates are brought slightly closer together and the material is deformed by the compression. The problem that will be considered here is the geometrical deformation of the material.

Let the distance between the two plates be w_0 and the displacement be δ. In the first instance, we consider a situation where the material between the plates is confined around the outer margin, as in a piston. Since the material cannot move laterally, it is restricted to the movement of compression, that is in the same direction as the displacement of one of the plates.



The location is measured relative to a point on the surface of the lower plate, $\mathbf{A_0}$. The location before the compression is $\boldsymbol{\lambda}_0$ and, after the compression, it is $\boldsymbol{\lambda}_1$. The difference between the locations is $\boldsymbol{\delta\lambda}$. This is expressed symbolically as follows.

$$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \boldsymbol{\delta\lambda} .$$

The problem comes down to writing an expression for the vector $\boldsymbol{\delta\lambda}$ as a function of location. In this situation the direction of the displacement is perpendicular to the two plates. Therefore, we construct a unit vector that is perpendicular to the bottom plate, $\mathbf{p}$. The part of $\boldsymbol{\lambda}_0$ that is relevant to the compression, $\boldsymbol{\delta\lambda}$, is the projection of $\boldsymbol{\lambda}_0$ upon $\mathbf{p}$, $\boldsymbol{\lambda} \bullet \mathbf{p}$, that is, the component of the location vector that is perpendicular to the lower plate. Overall, the material between the plates must compress a distance δ, but that compression is evenly distributed over the distance of $w = w_0 - |\delta|$. The perpendicular component of $\boldsymbol{\lambda}_0$, $\boldsymbol{\lambda} \bullet \mathbf{p}$, is divided by the interval between the plates, $w_0 - |\delta|$, and multiplied by the total amount of compression, δ, to give the amount of displacement of the location and that displacement is in the direction from the upper plate to the lower plate.

$$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \delta\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \delta * \frac{|\boldsymbol{\lambda}_0 \bullet \mathbf{p}|}{w_0 - |\delta|} * -\mathbf{p}.$$

Normally, the compression is small, on the order of 5%, or less. Most biological materials are relatively incompressible and inextensible largely because they are mostly water in colloidal gels or they are mineralized, as in bone. When it is important that biological materials do not compress, they are usually provided with a 'skeleton' of some sort or they are confined in an inextensible cavity.

If we were to look at the displacement vectors, they would start a vertical lines $\boldsymbol{\delta}$ units long at the upper plate and zero units long at the lower plate with a uniform gradation between the two.

If we look at the extension vectors, the vertical distance between two locations shortens a small amount, while the horizontal axes remain the same, so that there is a slight squashing or flattening of the test cube. If the vertical direction is the **z** axis, then the extension matrix is in the following form.

$$\mathcal{E} = [1.0, 1.0, 1.0 - \boldsymbol{\delta}].$$

The flattening is the same everywhere in the matrix, therefore the extension strain is uniform. There is no twisting or rotation in the matrix, so the orientation strain is a constant unity throughout the matrix.

$$\boldsymbol{O} = [1.0, 1.0, 1.0].$$

Unrestrained Compression

The constrained case of compression is simple and a bit boring, but it gives us a start on the unconstrained case, which is neither simple or boring. In the unconstrained we also assume an incompressible, or minimally compressible, amorphous matrix between two flat plates, but with room to move laterally, parallel with the plates. The gel will be assumed to be a circular mass, to simplify the calculations and because it is the lowest energy state for an unconstrained gel capable of flow.

When the plates move together, they compress the gel causing it to flow from the upper plate towards the lower plate, as in the example we was just considered. However, since the gel can flow laterally, it will do so and spread between the plates. We can say a few things about the movement of the gel, based upon the geometry of the situation.

First, the approximation of the plates by distance $|\boldsymbol{\delta}|$ will cause a volume, $\Delta V_{circle} = \pi r_0^2 * \delta$, of material to flow laterally from a circular region of radius r_0 about the center of the mass. The outer radius of the circular ring that contains the material that was inside the circular slab of radius r_0. is given by a simple expression.

$$\Delta V_{ring} = \pi r_1^2 (\boldsymbol{w}_0 - \delta) - \pi r_0^2 (\boldsymbol{w}_0 - \delta) = \pi (r_1^2 - r_0^2) w .$$

The volume of displaced material is equal to the volume of the ring that it occupies, so, we can set them equal and solve for the outer margin of the ring in terms of the vertical approximation and the radius of the original circular slab.

$$\pi r_0^2 * \delta = \pi (r_1^2 - r_0^2) w$$
$$r_0^2 * \delta = (r_1^2 - r_0^2) w$$
$$r_1^2 = \frac{r_0^2 w_0}{w_0 - \delta}$$
$$r_1 = r_0 * \sqrt{\frac{w_0}{w_0 - \delta}} .$$

Consequently, the central circular slab of radius r_0 now occupies a circular slab of radius r_1, where r_1 is given by the formula that was just derived. For small amounts of compression, the percentage increase in radius is about half the percentage decrease in height between the plates.

If the excess material were to flow uniformly peripherally, then the material at r_0 from the center of the disc would end up lying at r_1 from the center of the disc. However, flow is not uniform. It is most likely to flow as described by the Navier-Stokes Equation of hydrodynamics, that is laminar flow. If that is a reasonable approximation to the flow then the rate of flow is proportional to the square of the distance form the fixed walls. Close to the wall there is very little flow and near the plane midway between the two walls the flow would be maximal and proportional to the square of the distance to the nearest wall.

The lateral flow would be given by an expression like the following, where zero is taken to be the middle plane, half way between the two plates.

$$d(x) = d_{Max} - kx^2 .$$

The variable 'd' is the displacement of the material parallel to the plates and , 'x', is the distance to the nearest wall. The constant d_{Max} is the displacement of the material, in the middle horizontal plane, at x=0.0. We do not know d_{Max} or k, therefore we must solve for them in terms of the variables that we do know.

At the upper and lower plates the horizontal displacement is zero, which allows us to express the constant in terms of the maximal displacement .

$$d\left(\frac{w}{2}\right) = 0 = d_{Max} - kw^2 \quad \Leftrightarrow \quad k = \frac{d_{Max}}{\left(\frac{w}{2}\right)^2} = \frac{4\, d_{Max}}{w^2} .$$

Let $d_{Max} = \mu$, to simplify the notation a bit for calculation. Envision the flow as a stack of thin circular rings sheets that are expanding peripherally from the radius r_0 to the radius $r_0 + d(x)$, where d(x) is the diameter given by the expression for the displacement, and each sheet has a thickness of Δx. The volume of the expansion is the sum of those sheets, where each sheet has a volume that is the thickness of the sheet times its width.

$$V = \sum_{n=0}^{n=w/N} \pi * \left[(r_0 + d)^2 - r_0^2 \right] * \Delta x .$$

Passing to the limit for section thickness, dx, the expression can be written as an integral.

$$V = 2\int_0^{w/2} \pi * \left[(r_0 + d)^2 - r_0^2 \right] * dx$$
$$= 2\pi \int_0^{w/2} \left[(r_0 + d)^2 \right] * dx \text{ - } 2\pi r_0^2 \int_0^{w/2} 1 * dx \,.$$

The second integral evaluates as follows.

$$2\pi r_0^2 \int_0^{w/2} 1 \, dx = 2\pi r_0^2 \left[x \right]_0^{w/2} = \pi r_0^2 w$$

The first integral is more difficult to evaluate. The term within the square brackets can be expanded into the following expression.

$$\left(r_0 + \mu - \frac{4\mu}{w^2} x^2 \right)^2 = \frac{16\mu^2}{w^4} x^4 - 8\left(\frac{r_0 \mu}{w^2} + \frac{\mu^2}{w^2} \right) * x^2 + \left(\mu^2 + 2\mu r_0 + r_0^2 \right)$$

The integral of this expression turns out to be much simpler than one might expect.

$$2\pi \int_0^{w/2} \left[\frac{16\mu^2}{w^4} x^4 - 8\left(\frac{r_0 \mu}{w^2} + \frac{\mu^2}{w^2} \right) * x^2 + \left(\mu^2 + 2\mu r_0 + r_0^2 \right) \right] dx$$
$$= 2\pi \left[\frac{16\mu^2}{5w^4} x^5 - \frac{8}{3}\left(\frac{r_0 \mu}{w^2} + \frac{\mu^2}{w^2} \right) * x^3 + \left(\mu^2 + 2\mu r_0 + r_0^2 \right) * x \right]_0^{w/2}$$
$$= 2\pi \left[\frac{16\mu^2}{5w^4} \frac{w^5}{32} - \frac{8}{3}\left(\frac{r_0 \mu}{w^2} + \frac{\mu^2}{w^2} \right) * \frac{w^3}{8} + \left(\mu^2 + 2\mu r_0 + r_0^2 \right) * \frac{w}{2} \right]$$
$$= \pi \left[\frac{\mu^2}{5} - \frac{2}{3}\left(r_0 \mu + \mu^2 \right) + \left(\mu^2 + 2\mu r_0 + r_0^2 \right) \right] * w$$

If we subtract the second integral from the first, then the volume of the flow is given by the following expression.

$$V = \pi \left[\frac{8\mu^2}{15} + \frac{2}{3} r_0 \mu \right] * w$$

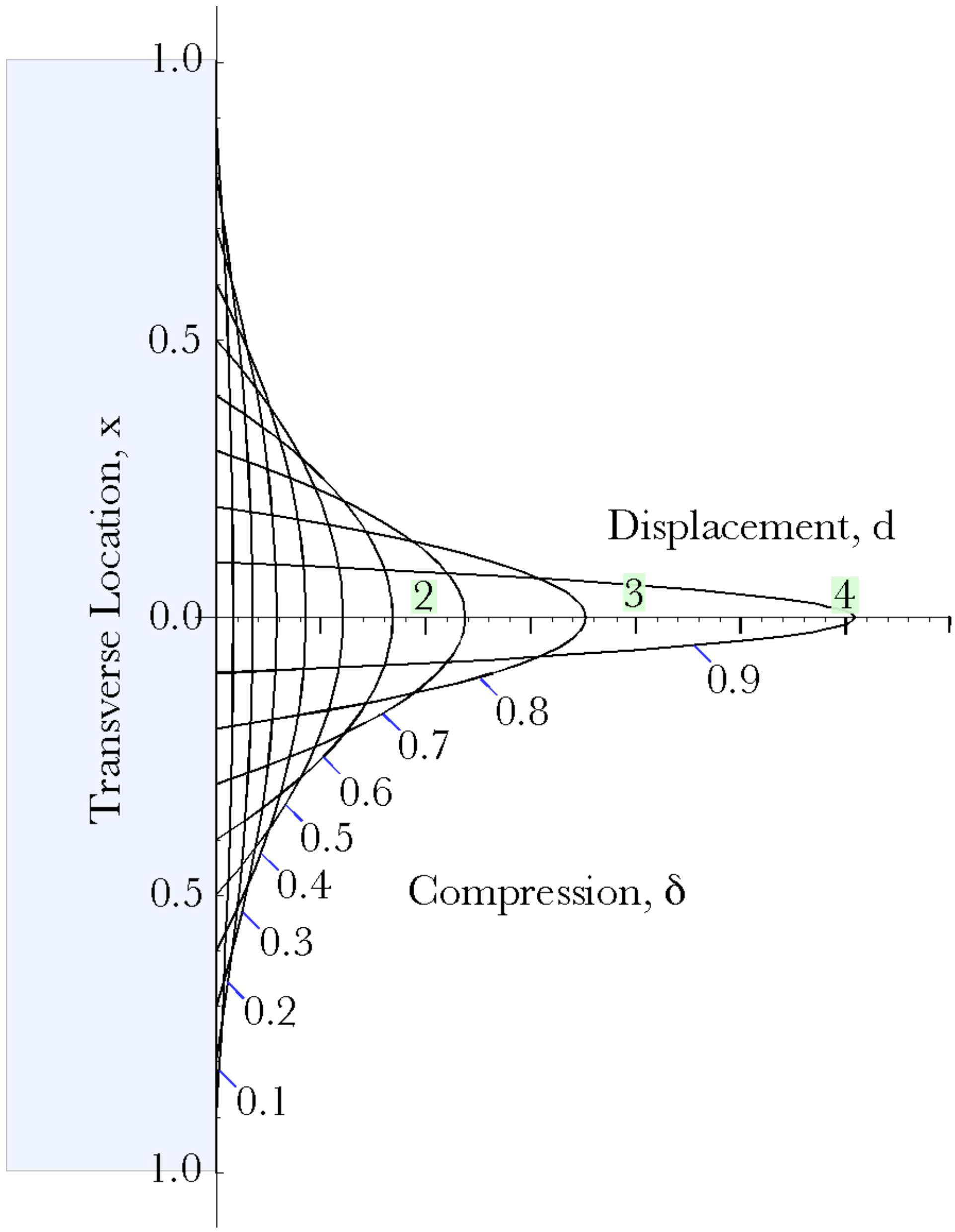
This expression can be set equal to the displaced volume and, with suitable rearrangement of the terms, it can be solved for the maximal displacement.

$$V = \pi\left[\frac{8\mu^2}{15} + \frac{2}{3}r_0\,\mu\right] * w = \pi\, r_0\,\delta \text{, therefore,}$$

$$\frac{8\mu^2 w}{15} + \frac{2}{3}r_0\,\mu w - r_0\,\delta = 0 \text{ and thus}$$

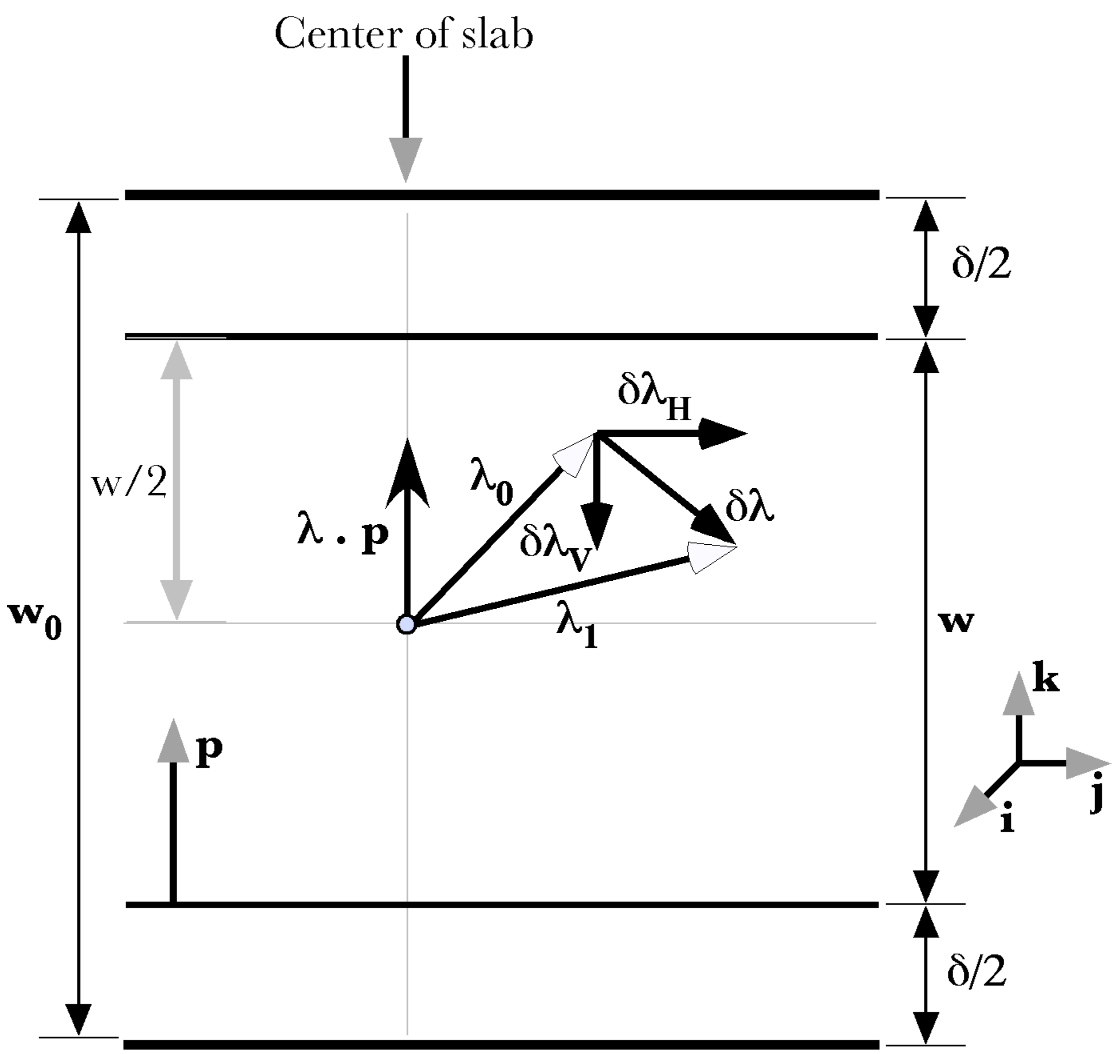
$$\mu = r_0\left[\sqrt{\left(\frac{5}{4}\right)^2 + \frac{15\delta}{8w}} - \frac{5}{4}\right].$$

The maximal displacement is a function of the radius of the circular slab, the amount of compression, and the distance between the plates.



The displacement profiles for different amounts of compression. The light blue area is a cross-section of a half of a circular slab of radius 1.0. The slab is compressed from 0.1 to 0.9 of its height (δ). This causes the material in the slab to be displaced peripherally (d). The volume of the displaced material is the displacement profile rotated in a circle about the center of the slab.

For small compressions, there is a small bulging of the matrix, approximately 8/10ths the percentage of the compression, so that a 10% compression causes the maximal lateral displacement to be about 8% of the radius of the slab. When the compression is 50%, the maximal displacement is about 60% of the slab radius. Finally, when the compression is large, like 90%, then the maximal displacement is large, 300% of the slab radius.



Compression without lateral constraint causes the matrix to move vertically and laterally. The vertical movement is greatest near the moving plates and the lateral movement is greatest near the middle horizontal plane.

The displacement vectors for compression vary in obliquity as a function of the transverse position of the location and the distance from the center of the slab. If we take the middle horizontal plane as our reference plane, then the locations in that plane are carried directly laterally a distance equal to the maximal displacement. At the bounding planes, the displacement is entirely vertical. Between those two extreme locations, the displacement is the sum of the vertical displacement, computed in the first example,

$$\boldsymbol{\Delta\lambda}_{\mathbf{V}} = \delta * \frac{|\boldsymbol{\lambda}_0 \bullet \mathbf{p}|}{w_0 - \delta} * -\mathbf{p},$$

and the lateral displacement for that horizontal plane, computed in this example

$$\boldsymbol{\Delta\lambda}_{\mathbf{H}} = \left(\mu - \frac{\mu}{w^2}x^2\right) * \frac{\mathbf{r}}{|\mathbf{r}|},$$

where $\mathbf{r}$ is the radial vector, from the center of the slab to the initial location and

$$\mu = r_0\left[\sqrt{\left(\frac{5}{4}\right)^2 + \frac{15\delta}{8w}} - \frac{5}{4}\right].$$

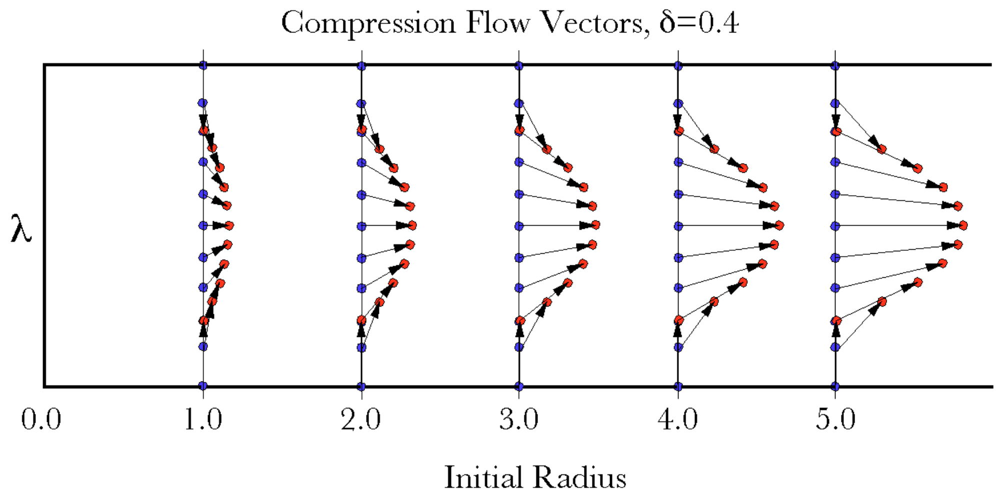
If the origin of the coordinate system is in the middle horizontal plane at the center of the slab and the initial location is $\boldsymbol{\lambda}_0 = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, then $\mathbf{r} = \alpha\mathbf{i} + \beta\mathbf{j}$. Therefore, assuming that the plates are parallel to the **i,j**-plane and equal distances above and below, that the compression brings them symmetrically towards the plane, and that the center of the slab is at the origin of the coordinate system, then the displacement vector is given by the following expression.

$$\begin{aligned}\boldsymbol{\Delta\lambda} = \boldsymbol{\Delta\lambda}_{\mathbf{V}} + \boldsymbol{\Delta\lambda}_{\mathbf{H}} &= \delta * \frac{|\boldsymbol{\lambda}_0 \bullet \mathbf{k}|}{w_0} * -\mathbf{k} + \left(\mu - \frac{\mu}{w^2}x^2\right) * \frac{\mathbf{r}}{|\mathbf{r}|} \\ &= \delta * \frac{-|\gamma\mathbf{k}|}{w_0} + \left(\mu - \frac{\mu}{w^2}x^2\right) * \frac{\alpha\mathbf{i} + \beta\mathbf{j}}{|\alpha\mathbf{i} + \beta\mathbf{j}|}.\end{aligned}$$

If we examine the positive **j,k**-plane, then the expression reduces to the following equation.

$$\boldsymbol{\Delta\lambda} = \mu\left(1 - \frac{x^2}{(w/2)^2}\right)\mathbf{j} - \delta * \frac{x}{(w/2)}\mathbf{k}, \text{ where } \mu = r_0\left[\sqrt{\left(\frac{5}{4}\right)^2 + \frac{15\delta}{8w}} - \frac{5}{4}\right].$$

For small amounts of compression, the lateral component is small. However, if the matrix is loosely constrained, like the nucleus pulposus of an intervertebral disc within the ligamentous sheath of the annulus fibrosus, then the lateral displacement may be the limiting parameter upon compression.



The flow vectors for compression between parallel plates are greater and more horizontal for middle levels and greater radii. The illustration is for a compression that is 0.4 times the distance between the plates. The blue markers indicate the original locations and the blue markers show where they flow.

In general, biological structures are not a neat as the computed examples considered here, but these examples do give reasonable models for the consideration of biological situations. While exact solutions of the biological geometry may not be possible, it is possible to generate order of magnitude solutions that support or deny particular interpretations.

The Extension Matrix

The basic formula for compression flow is the one given above and repeated here.

$$\boldsymbol{\Delta\lambda} = \boldsymbol{\Delta\lambda_V} + \boldsymbol{\Delta\lambda_H} = \mu\left(1 - \frac{x^2}{w^2}\right) * \frac{\alpha\mathbf{i} + \beta\mathbf{j}}{|\alpha\mathbf{i} + \beta\mathbf{j}|} + \delta * \frac{-|\gamma\mathbf{k}|}{w_0}.$$

To calculate the components of the extension matrix we need to substitute the locations that are slightly incremented and decremented in each of the cardinal directions and subtract the decremented value from the incremented value. The incremented location for the $\mathbf{i}$ direction is $(\alpha + \varepsilon)\mathbf{i} + \beta\,\mathbf{j}$. The difference clearly affects only the first term in the expression for $\boldsymbol{\Delta\lambda}$. The difference between the incremented and the decremented transformations is clearly a scalar factor times the increment.

$$\mu\left(1-\frac{x^2}{w^2}\right)*\frac{(\alpha+\varepsilon)\mathbf{i}+\beta\mathbf{j}}{|(\alpha+\varepsilon)\mathbf{i}+\beta\mathbf{j}|}-\mu\left(1-\frac{x^2}{w^2}\right)*\frac{(\alpha-\varepsilon)\mathbf{i}+\beta\mathbf{j}}{|(\alpha-\varepsilon)\mathbf{i}+\beta\mathbf{j}|}=\mu\left(1-\frac{x^2}{w^2}\right)*2\frac{\varepsilon\mathbf{i}}{|\varepsilon\mathbf{i}|}.$$

However, the maximal displacement (μ) term depends upon the distance from the center of the flow (r_0) and thus upon the **i** term.

$$\mu = r_0\left[\sqrt{\left(\frac{5}{4}\right)^2+\frac{15\delta}{8w}}-\frac{5}{4}\right]=r_0\kappa.$$

The radial distance is the square root of the squares of the **i** and **j** components of the location vector.

$$r_0=\sqrt{(\alpha\pm\varepsilon)^2+\beta^2}.$$

Consequently, the vector between the transformed increments depends upon the increment in a complex way.

$$\kappa\left(1-\frac{x^2}{w^2}\right)*\left(\sqrt{(\alpha+\varepsilon)^2+\beta^2}-\sqrt{(\alpha-\varepsilon)^2+\beta^2}\right)*2\frac{\varepsilon\mathbf{i}}{|\varepsilon\mathbf{i}|}.$$

The same arguments apply to the **j** axis. Therefore, we can write down its transformation by substituting the **j** increment into the expression in the place of the **i** increment.

$$\kappa\left(1-\frac{x^2}{w^2}\right)*\left(\sqrt{\alpha^2+(\beta+\varepsilon)^2}-\sqrt{\alpha^2+(\beta-\varepsilon)^2}\right)*2\frac{\varepsilon\mathbf{j}}{|\varepsilon\mathbf{j}|}.$$

The **k** axis is involved in the both terms of the equation for flow, therefore we can calculate the increment and decrement in much the same manner as the other two axes.

$$\left[\mu\left(1-\frac{(x+\varepsilon)^2}{w^2}\right)*\frac{\alpha\mathbf{i}+\beta\mathbf{j}}{|\alpha\mathbf{i}+\beta\mathbf{j}|}-\delta*\frac{|(x+\varepsilon)\mathbf{k}|}{w}\right]-\left[\mu\left(1-\frac{(x-\varepsilon)^2}{w^2}\right)*\frac{\alpha\mathbf{i}+\beta\mathbf{j}}{|\alpha\mathbf{i}+\beta\mathbf{j}|}-\delta*\frac{|(x-\varepsilon)\mathbf{k}|}{w}\right]$$

$$=\frac{\mu}{w^2}\left[(x-\varepsilon)^2-(x+\varepsilon)^2\right]*\frac{\alpha\mathbf{i}+\beta\mathbf{j}}{|\alpha\mathbf{i}+\beta\mathbf{j}|}-\delta*\frac{2|\varepsilon\mathbf{k}|}{w},$$

$$=-\frac{4\mu x\varepsilon}{\rho w^2}*(\alpha\mathbf{i}+\beta\mathbf{j})-\delta*\frac{2|\varepsilon\mathbf{k}|}{w},\ \text{where } \rho=\sqrt{\alpha^2+\beta^2}.$$

The components of the extension matrix are the ratio of the transformed increments to the untransformed increments.

$$\mathcal{E}(1) = \frac{\kappa\left(1-\frac{x^2}{w^2}\right)*\left(\sqrt{(\alpha+\varepsilon)^2+\beta^2}-\sqrt{(\alpha-\varepsilon)^2+\beta^2}\right)*2\frac{\varepsilon\mathbf{i}}{|\varepsilon\mathbf{i}|}\,.}{2\frac{\varepsilon\mathbf{i}}{|\varepsilon\mathbf{i}|}}$$

$$= \kappa\left(1-\frac{x^2}{w^2}\right)*\left(\sqrt{(\alpha+\varepsilon)^2+\beta^2}-\sqrt{(\alpha-\varepsilon)^2+\beta^2}\right);$$

$$\mathcal{E}(2) = \frac{\kappa\left(1-\frac{x^2}{w^2}\right)*\left(\sqrt{\alpha^2+(\beta+\varepsilon)^2}-\sqrt{\alpha^2+(\beta-\varepsilon)^2}\right)*2\frac{\varepsilon\mathbf{j}}{|\varepsilon\mathbf{j}|}}{2\frac{\varepsilon\mathbf{j}}{|\varepsilon\mathbf{j}|}}$$

$$= \kappa\left(1-\frac{x^2}{w^2}\right)*\left(\sqrt{\alpha^2+(\beta+\varepsilon)^2}-\sqrt{\alpha^2+(\beta-\varepsilon)^2}\right);$$

$$\mathcal{E}(3) = \frac{-\frac{4\mu x\varepsilon}{\rho w^2}*(\alpha\mathbf{i}+\beta\mathbf{j})-\delta*\frac{2|\varepsilon\mathbf{k}|}{w}}{2|\varepsilon\mathbf{k}|}$$

$$= -\frac{4\varepsilon^2\delta}{w}+\frac{8\mu\varepsilon^2 x}{w^2}\mathbf{i}-\frac{8\mu\varepsilon^2 x}{w^2}\mathbf{j}\,.$$

The matrix has two scalars and a quaternion, which means that the transform is an expansion or contraction in the **i,j**-plane and a rotation in the radial plane. Since the **i** and **j** terms in the quaternion are equal and the negative of each other, the axis of rotation is perpendicular to the radial vector.

$$\mathcal{E} = \left[\mathcal{E}(1),\quad \mathcal{E}(2),\quad \mathcal{E}(3)\right].$$

Therefore, we can write the formula for a location plus a small increment as follows.

$$\begin{aligned}\boldsymbol{\lambda}_1(\boldsymbol{\lambda}_0+\boldsymbol{\varepsilon}) &= \boldsymbol{\lambda}_1(\boldsymbol{\lambda}_0)+\boldsymbol{\varepsilon}+\mathcal{E}(\boldsymbol{\lambda}_0,\boldsymbol{\varepsilon})*\boldsymbol{\varepsilon}\,,\\ &= \boldsymbol{\lambda}_0+\boldsymbol{\varepsilon}+\boldsymbol{\Delta}\boldsymbol{\lambda}(\boldsymbol{\lambda}_0)+\mathcal{E}(\boldsymbol{\lambda}_0,\boldsymbol{\varepsilon})*\boldsymbol{\varepsilon}\,.\end{aligned}$$

The effect of an offset depends non-linearly upon the particular location (α,β), but is radially symmetrical in the **i,j**-plane. All the components depend on the **k** component of the location vector, because 'x' is the magnitude of the **k** component (γ).

Simplification of the extension matrix by the use of cylindrical coordinates

If we express the location in terms of cylindrical coordinates, (r,θ,z), then the third component of the extension matrix remains the same, but can be rewritten, the second component does not depend upon the size of the $\varepsilon\theta$, and it is only the first component can be simplified.

$$\Delta\boldsymbol{\lambda} = \Delta\boldsymbol{\lambda}_{\mathbf{H}} + \Delta\boldsymbol{\lambda}_{\mathbf{V}} = \mu\left(1 - \frac{x^2}{w^2}\right) * \mathbf{r} + \delta * \frac{-|\gamma\mathbf{k}|}{w_0},$$

$$\mu = r_0\left[\sqrt{\left(\frac{5}{4}\right)^2 + \frac{15\delta}{8w}} - \frac{5}{4}\right] = r_0\kappa \ .$$

We can write the expression for a radial offset as follows.

$$\kappa\left(1 - \frac{x^2}{w^2}\right) * \left((r+\varepsilon) - (r-\varepsilon)\right) = \kappa\left(1 - \frac{x^2}{w^2}\right) * 2\varepsilon,$$

$$\mathcal{E}(\mathbf{r}) = \frac{\kappa\left(1 - \frac{x^2}{w^2}\right) * 2\varepsilon}{2\varepsilon} = \kappa\left(1 - \frac{x^2}{w^2}\right).$$

Adding an offset to the angle will simply move the transform through that angular excursion, so $\mathcal{E}(\theta)$ is unity.

The vertical offset is the same, but it can be written in terms of the radial vector. If the radius is $\alpha\mathbf{i} + \beta\mathbf{j}$, then the axis of rotation is $\boldsymbol{\eta} = \beta\mathbf{i} - \alpha\mathbf{j}$, therefore the third component of the extension matrix can be written as follows.

$$\mathcal{E}(z) = -\frac{4\varepsilon^2\delta}{w} + \frac{8\mu\varepsilon^2 x}{w^2}\boldsymbol{\eta} = 4\varepsilon^2\left(-\frac{\delta}{w} + \frac{2\mu x}{w^2}\boldsymbol{\eta}\right)$$

The expression for the extension matrix is much simpler in cylindrical coordinates.

$$\mathcal{E} = \left[\kappa\left(1-\frac{x^2}{w^2}\right),\quad 1,\quad 4\varepsilon^2\left(-\frac{\delta}{w}+\frac{2\mu x}{w^2}\boldsymbol{\eta}\right)\right]\text{, therefore}$$

$$\boldsymbol{\lambda}_1(\boldsymbol{\lambda}_0+\boldsymbol{\varepsilon}) = \boldsymbol{\lambda}_1(\boldsymbol{\lambda}_0)+\boldsymbol{\varepsilon}+\mathcal{E}*\boldsymbol{\varepsilon}$$

$$= \boldsymbol{\lambda}_0+\boldsymbol{\Delta}\,\boldsymbol{\lambda}(\boldsymbol{\lambda}_0)+\boldsymbol{\varepsilon}+\left[\kappa\left(1-\frac{x^2}{w^2}\right)\quad 1\quad 4\varepsilon^2\left(-\frac{\delta}{w}+\frac{2\mu x}{w^2}\boldsymbol{\eta}\right)\right]*\begin{bmatrix}\varepsilon\bar{\mathbf{r}}\\ \varepsilon\theta\\ \varepsilon\bar{\mathbf{z}}\end{bmatrix}$$

The barred vectors are unit vectors in the directions **r** and **z**.

The Orientation Matrix

Since the **i** and **j** axes of the test cube remain in the same direction after the compression, independent of the location of the test cube the first component of the orientation matrix must be unity. The elements for the **y** and **z** elements are quaternions.

The transformed **i** axis is given by the following expression.

$$\kappa\left(1-\frac{x^2}{w^2}\right)*\left(\sqrt{(\alpha+\varepsilon)^2+\beta^2}-\sqrt{\alpha^2+\beta^2}\right)*\frac{\varepsilon\mathbf{i}}{|\varepsilon\mathbf{i}|}.$$

The transformation for the **j** axis is similar.

$$\kappa\left(1-\frac{x^2}{w^2}\right)*\left(\sqrt{\alpha^2+(\beta+\varepsilon)^2}-\sqrt{\alpha^2+\beta^2}\right)*\frac{\varepsilon\mathbf{j}}{|\varepsilon\mathbf{j}|}.$$

The transformation for the **k** axis is like the calculation for extension.

$$\left[\mu\left(1-\frac{(x+\varepsilon)^2}{w^2}\right)*\frac{\alpha\mathbf{i}+\beta\mathbf{j}}{|\alpha\mathbf{i}+\beta\mathbf{j}|}-\delta*\frac{|(x+\varepsilon)\mathbf{k}|}{w}\right]-\left[\mu\left(1-\frac{x^2}{w^2}\right)*\frac{\alpha\mathbf{i}+\beta\mathbf{j}}{|\alpha\mathbf{i}+\beta\mathbf{j}|}-\delta*\frac{|x\,\mathbf{k}|}{w}\right]$$

$$=\frac{\mu}{w^2}\left[x^2-(x+\varepsilon)^2\right]*\frac{\alpha\mathbf{i}+\beta\mathbf{j}}{|\alpha\mathbf{i}+\beta\mathbf{j}|}-\delta*\frac{|\varepsilon\mathbf{k}|}{w},$$

$$=-\frac{\mu\left(2x\varepsilon+\varepsilon^2\right)}{\rho w^2}*(\alpha\mathbf{i}+\beta\mathbf{j})-\delta*\frac{|\varepsilon\mathbf{k}|}{w},\text{ where }\rho=\sqrt{\alpha^2+\beta^2}.$$

The transformed **i** axis is a multiple of **i**, therefore the unit vector in the direction of the transformed **i** axis is simply **i**, $\hat{\mathbf{i}}=\mathbf{i}$. Similarly, **j** is the unit vector in the direction of the transformed **j** axis, $\hat{\mathbf{j}}=\mathbf{j}$. The **k** axis is more complex. The transformed **k** axis is given in the last set of equations.

$$-\frac{\mu\left(2x\varepsilon+\varepsilon^2\right)}{\rho w^2}*\left(\alpha\mathbf{i}+\beta\mathbf{j}\right)-\delta*\frac{|\varepsilon\mathbf{k}|}{w}$$

The magnitude of the vector is given by the following expression.

$$\upsilon=\sqrt{2\left(\frac{\mu\left(2x\varepsilon+\varepsilon^2\right)}{\rho w^2}\right)^2+\left(\frac{\delta\varepsilon}{w}\right)^2}$$

The unit vector in the direction of the transformed **k** axis is the transformed axis over its magnitude.

$$\hat{\mathbf{k}}=-\frac{\mu\left(2x\varepsilon+\varepsilon^2\right)}{\upsilon\rho w^2}\alpha\mathbf{i}-\frac{\mu\left(2x\varepsilon+\varepsilon^2\right)}{\upsilon\rho w^2}\beta\mathbf{j}-\frac{\delta\varepsilon}{\upsilon w}\mathbf{k}$$

We can replace the scalar multiples of the basis vectors with new symbols to simplify the calculations.

$$\hat{\mathbf{k}}=-\hat{\alpha}\,\mathbf{i}-\hat{\beta}\,\mathbf{j}-\hat{\gamma}\,\mathbf{k}$$

The frame for the $\hat{\mathbf{i}}$ axis is clearly $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$, since $\hat{\mathbf{i}}$ is orthogonal to $\hat{\mathbf{j}}$ and the $\overline{\mathbf{z}}$ axis is irrelevant to that calculation.

$$\boldsymbol{Q}\left(\boldsymbol{f}_{\mathrm{x}}\right)=1.0\,.$$

The frame for $\hat{\mathbf{j}}$ is more difficult to obtain.

$$\boldsymbol{Q}_{yz}=\frac{-\hat{\alpha}\,\mathbf{i}-\hat{\beta}\,\mathbf{j}-\hat{\gamma}\,\mathbf{k}}{\mathbf{j}}=\left(-\hat{\alpha}\,\mathbf{i}-\hat{\beta}\,\mathbf{j}-\hat{\gamma}\,\mathbf{k}\right)*\left(-\mathbf{j}\right)=-\hat{\beta}-\hat{\gamma}\,\mathbf{i}+\hat{\alpha}\,\mathbf{k}\ ;$$

$$\zeta_{\boldsymbol{Q}}=\sqrt{\hat{\alpha}^2+\hat{\beta}^2+\hat{\gamma}^2}=1.0,\ \text{because}\ -\hat{\alpha}\,\mathbf{i}-\hat{\beta}\,\mathbf{j}-\hat{\gamma}\,\mathbf{k}\ \text{is a unit vector.}$$

The unit vector of the rotation quaternion is given by the following expression.

$$\frac{\mathcal{V}\left(\boldsymbol{Q}_{yz}\right)}{\zeta_{\mathbf{V}}}=\frac{-\hat{\gamma}\,\mathbf{i}+\hat{\alpha}\,\mathbf{k}}{\sqrt{\hat{\alpha}^2+\hat{\gamma}^2}}\,.$$

Consequently, the projected **z** axis is computed as follows.

$$\overline{\mathbf{z}}=\boldsymbol{q}_{yz}\left(\frac{\pi}{4}\right)*\hat{\mathbf{y}}*\boldsymbol{q}_{yz}\left(\frac{\pi}{4}\right)^{-1}=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\frac{-\hat{\gamma}\,\mathbf{i}+\hat{\alpha}\,\mathbf{k}}{\sqrt{\hat{\alpha}^2+\hat{\gamma}^2}}\right)*\mathbf{j}*\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\frac{\hat{\gamma}\,\mathbf{i}-\hat{\alpha}\,\mathbf{k}}{\sqrt{\hat{\alpha}^2+\hat{\gamma}^2}}\right)$$

$$=\frac{-\hat{\alpha}\mathbf{i}-\hat{\gamma}\mathbf{k}}{\sqrt{\hat{\alpha}^2+\hat{\gamma}^2}}\,.$$

Finally, the **x** axis is the vector of the rotation quaternion.

$$\bar{\mathbf{x}} = \frac{-\hat{\gamma}\mathbf{i} + \hat{\alpha}\mathbf{k}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}}.$$

The frame for the $\bar{\mathbf{y}}$ vector is given by the computed set of vectors.

$$f_{\bar{y}} = \left\{\bar{\mathbf{x}}_{\bar{y}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}_{\bar{y}}\right\} = \left\{\frac{-\hat{\gamma}\mathbf{i} + \hat{\alpha}\mathbf{k}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}}, \quad \mathbf{j}, \quad \frac{-\hat{\alpha}\mathbf{i} - \hat{\gamma}\mathbf{k}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}}\right\}$$

The second component of the orientation matrix is the ratio of this frame of reference to the frame of reference $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. To calculate the orientation quaternion for these two frames it is simpler to note that the $\bar{\mathbf{y}}$ is $\mathbf{j}$, which is the same as in the unrotated frame of reference. The two frames are aligned at that axis. All that remains is to determine what rotation will rotate the unstrained $\mathbf{x}$ axis into $\bar{\mathbf{x}}$. That is obtained by dividing $\bar{\mathbf{x}}$ by $\mathbf{x}$.

$$Q(\mathbf{x} \to \bar{\mathbf{x}}) = \frac{\dfrac{-\hat{\gamma}\mathbf{i} + \hat{\alpha}\mathbf{k}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}}}{\mathbf{i}} = \frac{-\hat{\gamma}\mathbf{i} + \hat{\alpha}\mathbf{k}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}} * -\mathbf{i} = \frac{-\hat{\gamma} - \hat{\alpha}\mathbf{j}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}},$$

$$Q(f_y) = \frac{-\hat{\gamma} - \hat{\alpha}\mathbf{j}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}}.$$

This means that the original, pre-strain, frame of reference is rotated about the $\mathbf{z}$ axis through an angle ϕ, which is given by the following expression.

$$\phi_{\bar{y}} = cos^{-1}\left(\frac{-\hat{\gamma}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}}\right).$$

Summing up, the change in orientation of the frame of reference for the $\bar{\mathbf{y}}$ component is a rotation of $\phi_{\bar{y}}$ about the $\mathbf{y}$ axis.

The frame for the $\bar{\mathbf{z}}$ component is obtained in a similar manner. The $\bar{\mathbf{z}}$ component is the unit vector, $\hat{\alpha}\mathbf{i} + \hat{\beta}\mathbf{j} + \hat{\gamma}\mathbf{k}$. The rotation quaternion that rotates $\bar{\mathbf{z}}$ into $\bar{\mathbf{x}}$ is given by the following expression.

$$Q_{zx} = \frac{\mathbf{i}}{-\hat{\alpha}\mathbf{i} - \hat{\beta}\mathbf{j} - \hat{\gamma}\mathbf{k}} = \mathbf{i}\left(\hat{\alpha}\mathbf{i} + \hat{\beta}\mathbf{j} + \hat{\gamma}\mathbf{k}\right) = -\hat{\alpha} - \hat{\gamma}\mathbf{j} + \hat{\beta}\mathbf{k}\,;$$

$$\zeta_Q = \sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2} = 1.0, \text{ because } -\hat{\alpha}\mathbf{i} - \hat{\beta}\mathbf{j} - \hat{\gamma}\mathbf{k} \text{ is a unit vector.}$$

The unit vector of the rotation quaternion is given by the following expression.

$$\frac{\mathcal{V}(\boldsymbol{Q}_{yz})}{\zeta_{\mathbf{V}}} = \frac{-\hat{\gamma}\,\mathbf{j} + \hat{\beta}\,\mathbf{k}}{\sqrt{\hat{\beta}^2 + \hat{\gamma}^2}} = \overline{\mathbf{y}}\,.$$

Consequently, the projected **x** axis is computed by rotating $\overline{\mathbf{z}}$ though 90° about the vector of the rotation quaternion.

$$\overline{\mathbf{x}} = \boldsymbol{q}_{\mathrm{zx}}\left(\frac{\pi}{4}\right) * \hat{\mathbf{z}} * \boldsymbol{q}_{\mathrm{zx}}\left(\frac{\pi}{4}\right)^{-1} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{-\hat{\gamma}\,\mathbf{j} + \hat{\beta}\,\mathbf{k}}{\sqrt{\hat{\beta}^2 + \hat{\gamma}^2}}\right) * \left(-\hat{\alpha}\,\mathbf{i} - \hat{\beta}\,\mathbf{j} - \hat{\gamma}\,\mathbf{k}\right) * \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{\hat{\gamma}\,\mathbf{j} - \hat{\beta}\,\mathbf{k}}{\sqrt{\hat{\beta}^2 + \hat{\gamma}^2}}\right)$$

$$= \frac{\left(\hat{\beta}^2 + \hat{\gamma}^2\right)\mathbf{i} - \hat{\alpha}\hat{\beta}\mathbf{j} - \hat{\alpha}\hat{\gamma}\mathbf{k}}{\sqrt{\hat{\beta}^2 + \hat{\gamma}^2}}\,.$$

Finally, as noted above, the **y** axis is the vector of the rotation quaternion.

$$\overline{\mathbf{y}} = \frac{-\hat{\gamma}\,\mathbf{j} + \hat{\beta}\,\mathbf{k}}{\sqrt{\hat{\beta}^2 + \hat{\gamma}^2}}$$

The frame for the $\overline{\mathbf{z}}$ vector is given by the computed set of vectors.

$$\boldsymbol{f}_{\overline{\mathbf{z}}} = \left\{\overline{\mathbf{x}}_{\overline{\mathbf{z}}}, \overline{\mathbf{y}}_{\overline{\mathbf{z}}}, \overline{\mathbf{z}}\right\} = \left\{\frac{\left(\hat{\beta}^2 + \hat{\gamma}^2\right)\mathbf{i} - \hat{\alpha}\hat{\beta}\mathbf{j} - \hat{\alpha}\hat{\gamma}\mathbf{k}}{\sqrt{\hat{\beta}^2 + \hat{\gamma}^2}},\quad \frac{-\hat{\gamma}\,\mathbf{j} + \hat{\beta}\,\mathbf{k}}{\sqrt{\hat{\beta}^2 + \hat{\gamma}^2}},\quad -\hat{\alpha}\,\mathbf{i} - \hat{\beta}\,\mathbf{j} - \hat{\gamma}\,\mathbf{k}\right\}$$

This looks complex, but a little algebraic manipulation will confirm that all the vectors are unit vectors and the computation was designed to make sure that they are mutually orthogonal.

The third component of the orientation matrix is the ratio of this frame of reference to the pre-strain frame of reference $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. It is not as straight-forward to calculate the orientation quaternion for these two frames as it was for $\overline{\mathbf{y}}$. Both the swing and spin quaternions must be calculated. We start by calculating the rotation that will swing the pre-strain **z** axis (**k**) into the post-strain **z** axis ($-\hat{\alpha}\,\mathbf{i} - \hat{\beta}\,\mathbf{j} - \hat{\gamma}\,\mathbf{k}$).

$$\boldsymbol{Q}\left(\mathbf{z} \rightarrow \overline{\mathbf{z}}\right) = \frac{-\hat{\alpha}\,\mathbf{i} - \hat{\beta}\,\mathbf{j} - \hat{\gamma}\,\mathbf{k}}{\mathbf{k}} = -\hat{\alpha}\,\mathbf{i} - \hat{\beta}\,\mathbf{j} - \hat{\gamma}\,\mathbf{k} * -\mathbf{k} = -\hat{\gamma} + \hat{\beta}\,\mathbf{i} - \hat{\alpha}\,\mathbf{j}\,.$$

This means that the original, pre-strain, frame of reference is rotated about the vector $\hat{\beta}\,\mathbf{i} - \hat{\alpha}\,\mathbf{j}$ through an angle ϕ_{Sw}, which is given by the following expression.

$$\phi_{\mathrm{Sw}} = cos^{-1}\left(-\hat{\gamma}\right).$$

We can now apply the same rotation to the other two axes of the pre-strain frame of reference to obtain the intermediate frame.

$$\tilde{\mathbf{x}} = \frac{1}{\sqrt{2}}\left(1+\frac{\hat{\beta}\,\mathbf{i}-\hat{\alpha}\,\mathbf{j}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\right)*\mathbf{i}*\frac{1}{\sqrt{2}}\left(1+\frac{-\hat{\beta}\,\mathbf{i}+\hat{\alpha}\,\mathbf{j}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\right)=\frac{\hat{\beta}^2}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{i}-\frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{j}+\frac{\hat{\alpha}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\mathbf{k}\,;$$

$$\tilde{\mathbf{y}} = \frac{1}{\sqrt{2}}\left(1+\frac{\hat{\beta}\,\mathbf{i}-\hat{\alpha}\,\mathbf{j}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\right)*\mathbf{j}*\frac{1}{\sqrt{2}}\left(1+\frac{-\hat{\beta}\,\mathbf{i}+\hat{\alpha}\,\mathbf{j}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\right)=\frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{i}-\frac{\hat{\alpha}^2}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{j}+\frac{\hat{\beta}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\mathbf{k}\,.$$

We need to select a final axis other than $\bar{\mathbf{z}}$ and compute the swing that rotates the intermediate axis into the final axis. In this case the $\bar{\mathbf{y}}$ axis looks like it will involve less calculation.

$$\boldsymbol{Q}(\tilde{\mathbf{y}}\to\bar{\mathbf{y}}) = \frac{\dfrac{-\hat{\gamma}\,\mathbf{j}+\hat{\beta}\mathbf{k}}{\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}}{\dfrac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{i}-\dfrac{\hat{\alpha}^2}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{j}+\dfrac{\hat{\beta}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\mathbf{k}} = \frac{-\hat{\gamma}\,\mathbf{j}+\hat{\beta}\mathbf{k}}{\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}*\left(-\frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{i}+\frac{\hat{\alpha}^2}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{j}-\frac{\hat{\beta}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\mathbf{k}\right)$$

$$=\left(\frac{\hat{\alpha}^2\hat{\gamma}}{\left(\hat{\alpha}^2+\hat{\beta}^2\right)\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}+\frac{\hat{\beta}^2}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}\right)+\left(\frac{\hat{\beta}\hat{\gamma}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}-\frac{\hat{\beta}\hat{\alpha}^2}{\hat{\alpha}^2+\hat{\beta}^2\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}\right)\mathbf{i}$$

$$-\frac{\hat{\alpha}\hat{\beta}^2}{\left(\hat{\alpha}^2+\hat{\beta}^2\right)\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}\mathbf{j}-\frac{\hat{\alpha}\hat{\beta}\hat{\gamma}}{\left(\hat{\alpha}^2+\hat{\beta}^2\right)\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}\mathbf{k}\,.$$

Since the both vectors in the ratio are unit vectors, the ratio must be a unit quaternion and thus we can state that the scalar component of the ratio is the cosine of the angle of rotation.

$$cos\,\phi_{\mathrm{Sp}} = \left(\frac{\hat{\alpha}^2\hat{\gamma}}{\left(\hat{\alpha}^2+\hat{\beta}^2\right)\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}+\frac{\hat{\beta}^2}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}\sqrt{\hat{\beta}^2+\hat{\gamma}^2}}\right).$$

If we examine the two vectors in the ratio, it may be noted that they can be written in the following forms.

$$\frac{-\hat{\gamma}\,\mathbf{j}+\hat{\beta}\mathbf{k}}{\sqrt{\hat{\beta}^2+\hat{\gamma}^2}} = cos\,\varphi_\gamma\mathbf{j}+sin\,\varphi_\gamma\mathbf{k} \text{ and}$$

$$\frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{i}-\frac{\hat{\alpha}^2}{\hat{\alpha}^2+\hat{\beta}^2}\mathbf{j}+\frac{\hat{\beta}}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}}\mathbf{k} = sin\,\varphi_\alpha\,cos\,\varphi_\alpha\mathbf{i}-cos^2\,\varphi_\alpha\mathbf{j}+sin\,\varphi_\alpha\mathbf{k}\,.$$

This leads to the scalar term in the ratio being a function of the two angles.

$$cos\,\phi_{Sp} = cos^2\varphi_\alpha\, cos\varphi_\gamma + sin\varphi_\alpha\, sin\varphi_\gamma.$$

In the trigonometric notation the vector for the axis of rotation is written as follows.

$$\mathbf{v_r} = \left(cos\varphi_\gamma\, sin\varphi_\alpha - sin\varphi_\gamma\, cos^2\varphi_\alpha\right)\mathbf{i} - sin\varphi_\gamma\, sin\varphi_\alpha\, cos\varphi_\alpha\,\mathbf{j} - cos\varphi_\gamma\, sin\varphi_\alpha\, cos\varphi_\alpha\,\mathbf{k}\,.$$

The symmetry of the equation is more apparent in the trigonometric notation. It may also make another relationship more explicit. The φ_γ angle arose from the swing rotation that aligned the $\mathbf{z}$ axis with the $\bar{\mathbf{z}}$ axis. It is the angle of the vector that is the axis of rotation for the swing. The $\varphi_{\boldsymbol{\alpha}}$ angle is the angle of the intermediate form of the $\mathbf{y}$ axis vector, $\hat{\mathbf{y}}$, the $\mathbf{y}$ axis after it has experienced the swing, but before the spin transformation.

The rotation quaternion for the transformation of the **z** frame of reference relative to the pre-strain frame of reference is the combination of the swing transformation and then the spin transformation.

$$\begin{aligned} \boldsymbol{Q}(\boldsymbol{f}_\mathbf{z}) &= \begin{bmatrix} \left(cos^2\varphi_\alpha\, cos\varphi_\gamma + sin\varphi_\alpha\, sin\varphi_\gamma\right) + \left(cos\varphi_\gamma\, sin\varphi_\alpha - sin\varphi_\gamma\, cos^2\varphi_\alpha\right)\mathbf{i} \\ - sin\varphi_\gamma\, sin\varphi_\alpha\, cos\varphi_\alpha\,\mathbf{j} - cos\varphi_\gamma\, sin\varphi_\alpha\, cos\varphi_\alpha\,\mathbf{k} \end{bmatrix} * \left[-\hat{\gamma} + \hat{\beta}\,\mathbf{i} - \hat{\alpha}\,\mathbf{j}\right], \\ &= \left(\omega + \xi\mathbf{i} + \psi\mathbf{j} + \zeta\mathbf{k}\right) * \left(-\hat{\gamma} + \hat{\beta}\,\mathbf{i} - \hat{\alpha}\,\mathbf{j}\right), \\ &= -\left(\hat{\gamma}\omega + \hat{\beta}\xi + \hat{\alpha}\psi\right) + \left(\hat{\beta}\omega - \hat{\gamma}\xi - \hat{\alpha}\zeta\right)\mathbf{i} + \left(-\hat{\alpha}\omega + \hat{\gamma}\psi - \hat{\beta}\zeta\right)\mathbf{j} + \left(-\hat{\alpha}\xi + \hat{\beta}\psi + \hat{\gamma}\zeta\right)\mathbf{k}\,. \end{aligned}$$

If this expression is written out in all of the original terms that we started with, it is a horrendous expression. However, a great deal of the complication comes from the fact that we have kept the analysis in symbolic terms. If one is doing the calculation with numerical values the last expression is not much more difficult than any of the other expressions for rotation quaternions. In the end, we always come down to a quaternion with four terms, which represents a rotation of a particular angular excursion about a particular vector.

We have computed the rotation quaternions for all three frames of reference. Consequently, we can write the orientation matrix.

$$\boldsymbol{O} = \left[\boldsymbol{Q}(\boldsymbol{f}_{\mathbf{x}}), \boldsymbol{Q}(\boldsymbol{f}_{\mathbf{y}}), \boldsymbol{Q}(\boldsymbol{f}_{\mathbf{z}})\right]$$

$$= \left[1.0, \frac{-\hat{\gamma} - \hat{\alpha}\,\mathbf{j}}{\sqrt{\hat{\alpha}^2 + \hat{\gamma}^2}}, -\left(\hat{\gamma}\omega + \hat{\beta}\xi + \hat{\alpha}\psi\right) + \left(\hat{\beta}\omega - \hat{\gamma}\xi - \hat{\alpha}\zeta\right)\mathbf{i} + \left(-\hat{\alpha}\omega + \hat{\gamma}\psi - \hat{\beta}\zeta\right)\mathbf{j} + \left(-\hat{\alpha}\xi + \hat{\beta}\psi + \hat{\gamma}\zeta\right)\mathbf{k}\right]$$