Extension and Orientation Strain

The attribute that we are trying to capture when creating indices of extension and orientation strain is the squeezing and twisting the happens when a matrix is compressed or stretched, sheared or wrenched. We have a sense of what is happening when the matrix is strained, but we must reduce that feeling to a consistently defined quantity that can be manipulated to extract information about what is occurring in the matrix. If a small cube of the matrix is selected, then the distortion causes the sides to grow, or shrink, to rotate relative to each other, and/or to change their shape. We wish to somehow attach numbers to those changes in a meaningful way.

If we are dealing with a nearly incompressible and inextensible matrix, which is often approximately the situation with biological materials, because they flow, then volume will stay nearly the same. Since the material of the matrix flows, any local increase or decrease of pressure will cause material to flow out of or into the region. Generally, as one dimension is reduced, another increases. If a cube is compressed to be half as high then it must become about 40% wider ($\sqrt{2}-1$). Putting a tensile force through the matrix will lengthen in the direction of the force, but narrow in the orthogonal plane.

Biological matrices often contain proteins that resist distension and which will draw the matrix back into a particular conformation, if stresses are removed. Some biological matrices are fluids and they flow to a minimal energy state. Cartilage is a mixture of a proteoglycan matrix and an aqueous fluid that may shift relative to each other when compressed and decompressed.

Location Strain

Strains in materials may be viewed from several perspectives. Perhaps simplest, is that the matrix moves or flows. That is to say, there is a change in location. If we place a small marker at some point in the matrix ($\boldsymbol{\lambda}_0$) prior to the strain and observe its location after the strain is imposed ($\boldsymbol{\lambda}_1$), then there is a change of location ($\boldsymbol{\Delta\lambda}$) that usually depends upon the original location.

$$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \boldsymbol{\Delta\lambda}\,.$$

The description of $\boldsymbol{\Delta\lambda}$ is the major part of characterizing location strain. The form of $\boldsymbol{\Delta\lambda}$ is characteristic of each type of strain. Particular derivations of $\boldsymbol{\Delta\lambda}$ are treated elsewhere.

Extension Strain

Let us start with a cube of material, embedded in the matrix. Each of its corners has a location prior to the distortion. If we pick the center of the cube, $\boldsymbol{\lambda}_C = \{\boldsymbol{\lambda}_x, \boldsymbol{\lambda}_y, \boldsymbol{\lambda}_z\}$, as the location of the cube, then the corners can be written as $\{\boldsymbol{\lambda}_x \pm \boldsymbol{\varepsilon}\,\mathbf{x}, \boldsymbol{\lambda}_y \pm \boldsymbol{\varepsilon}\,\mathbf{y}, \boldsymbol{\lambda}_z \pm \boldsymbol{\varepsilon}\,\mathbf{z}\}$, where all the $\boldsymbol{\varepsilon}$'s are non-zero and of equal magnitude. If the location transformation is applied to each corner, then the resulting box will give an indication of the local distortion. Consequently, a comparison of the box before and after the strain is imposed might serve to illustrate the local change in the metric of the matrix. Such visual comparisons are excellent for obtaining an intuitive feel for the nature of the transformation, but they are not very useful in calculation, except in guiding the flow of the calculation and checking the results. There must also be quantitative measurements, indices that encapsulate the distortion and allow one to calculate the consequences of the strain.

The distortion of extension may be encapsulated in a matrix of three rotation quaternions. There are a number of ways that one might compute the local extension strain. However, to be definite, let the transformed axes be the three vectors between opposite faces of the distorted cube, $\{\boldsymbol{\lambda}_x \pm \boldsymbol{\varepsilon}\,\mathbf{x}, \boldsymbol{\lambda}_y, \boldsymbol{\lambda}_z\}$, $\{\boldsymbol{\lambda}_x, \boldsymbol{\lambda}_y \pm \boldsymbol{\varepsilon}\,\mathbf{y}, \boldsymbol{\lambda}_z\}$, and $\{\boldsymbol{\lambda}_x, \boldsymbol{\lambda}_y, \boldsymbol{\lambda}_z \pm \boldsymbol{\varepsilon}\,\mathbf{z}\}$. The vector along the $\mathbf{x}$ axis will be $\boldsymbol{\alpha}_x = \{\boldsymbol{\lambda}_x + \boldsymbol{\varepsilon}\,\mathbf{x}, \boldsymbol{\lambda}_y, \boldsymbol{\lambda}_z\} - \{\boldsymbol{\lambda}_x - \boldsymbol{\varepsilon}\,\mathbf{x}, \boldsymbol{\lambda}_y, \boldsymbol{\lambda}_z\}$. Similar expressions apply to the vectors along the $\mathbf{y}$ axis, $\boldsymbol{\alpha}_y$, and the $\mathbf{z}$ axis, $\boldsymbol{\alpha}_z$. The rotation that transforms the $\mathbf{x}$ axis pre-strain, $\mathbf{x}$, into the $\mathbf{x}$ axis post-strain, $\boldsymbol{\alpha}_x$, is the ratio of the post-strain axis to the pre-strain axis.

$$\boldsymbol{q}_x = \frac{\boldsymbol{\alpha}_x}{\mathbf{x}}.$$

Similar expressions can be written of the transformations of the $\mathbf{y}$ and $\mathbf{z}$ axes.

This leads to the expression for the transformation of the local coordinates, the extension strain.

$$\boldsymbol{\tau}(\mathbf{V}) = \begin{bmatrix} \boldsymbol{q}_x & \boldsymbol{q}_y & \boldsymbol{q}_z \end{bmatrix} * \begin{bmatrix} \mathbf{x}_\mathrm{V} \\ \mathbf{y}_\mathrm{V} \\ \mathbf{z}_\mathrm{V} \end{bmatrix}, \text{ where } \mathbf{V} = \mathbf{x}_\mathrm{V} + \mathbf{y}_\mathrm{V} + \mathbf{z}_\mathrm{V}$$

$$\boldsymbol{\tau}(\mathbf{V}) = \boldsymbol{q}_x * \mathbf{x}_\mathrm{V} + \boldsymbol{q}_y * \mathbf{y}_\mathrm{V} + \boldsymbol{q}_z * \mathbf{z}_\mathrm{V}$$

The new vector is the sum of the quaternion products of its components. If we write the extension transformation as $\mathcal{E}_Q = \begin{bmatrix} \boldsymbol{q}_x & \boldsymbol{q}_y & \boldsymbol{q}_z \end{bmatrix}$, then the vector transformation may be written as follows.

$$\mathbf{V}' = \mathcal{E}_Q * \mathbf{V}\,.$$

It should be noted that there are other options for the strain vectors. For instance, we might have chosen to make the location one corner of the box and used the three edges of its cube as the axes. The calculation would be otherwise similar.

Orientation Strain

Orientation is defined relative to orthogonal frames of reference. It could be defined otherwise, but a squashed frame of reference cannot be the result of a single rotation of an orthogonal frame of reference. Consequently, it is necessary to define a set of mutually orthogonal unit vectors that encapsulate the orientation of a region of the matrix, even when it has been squashed. Pre-strain one can define the orientation any way that one wants, but it is generally convenient to choose a set of vectors that encapsulate some attribute of the system. For instance, choosing the primary vector perpendicular to a fixed plate or radially directed from a depression. The choice of the other two vectors within the plane perpendicular to the primary vector may be arbitrary or may reflect another pertinent feature of the geometry.

When the vectors are specified , then the transformation of the location plus a small displacement in the direction of one of the frame vectors ($\boldsymbol{\varepsilon}_w$, $\boldsymbol{w} = \boldsymbol{x}, \boldsymbol{y}.\boldsymbol{z}$) is computed and the transformed location is subtracted from the transformed location. The unit vector of the difference is the transformed coordinate axis.

$$\begin{aligned}
&\boldsymbol{\lambda}(\boldsymbol{\lambda}_0) = \boldsymbol{\lambda}_0 + \Delta\,\boldsymbol{\lambda}(\boldsymbol{\lambda}_0)\,;\\
&\boldsymbol{\lambda}(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}_w) = \boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}_w + \Delta\,\boldsymbol{\lambda}(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}_w)\,;\\
&\mathbf{w}' = \boldsymbol{\lambda}(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}_w) - \boldsymbol{\lambda}(\boldsymbol{\lambda}_0)\,;\\
&\overline{\mathbf{w}} = \frac{\boldsymbol{\lambda}(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}_w) - \boldsymbol{\lambda}(\boldsymbol{\lambda}_0)}{\left|\boldsymbol{\lambda}(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}_w) - \boldsymbol{\lambda}(\boldsymbol{\lambda}_0)\right|}\,.
\end{aligned}$$

Properly speaking, the calculation should be done using limits as $\boldsymbol{\varepsilon}_w$ approaches zero, but, if we make it small relative to the magnitude of the transformation of the location, the result should be a reasonable estimate of the direction of the coordinate axis. The three direction unit vectors ($\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}$) are the basis for computing the orientation frames for the three transformed axes.

Post-strain the original frame has been distorted by the strain and it is desirable to indicate the change in orientation as the ratio of the post-strain orientation frame to the pre-strain orientation frame. However, if the strain has carried the mutually orthogonal axes out of orthogonality, there is not a single rotation that will carry the pre-strain frame into the post-strain frame. A means needs to be found to deal with the loss of orthogonality.

It is convenient to retain the concept of the orientation as a plane with a perpendicular vector, but there are a number of ways that such a construction might be obtained. Again, for definiteness, let the orientation of the transformed **x** axis be the new **x** axis and the plane perpendicular to it. The same approach is used for the other two axes. However, we are still left with determining the directions of the transformed **y** and **z** axes in the plane perpendicular to the new **x** axis. We can obtain the projections of the **y** and **z** axes into the plane perpendicular to the **x** axis by rotating the **x** axis 90° towards the **y** axis and using the axis of rotation as the **z** axis. In that situation, the planar **y** axis is the ratio of the **y** axis to the **x** axis with an angle of 90° and a tensor of 1.0. times the **x** unit vector.

$$\bar{\mathbf{x}} = \frac{\mathbf{x}'}{|\mathbf{x}'|} ;$$

$$Q_{xy} = \frac{\mathbf{y}'/|\mathbf{y}'|}{\bar{\mathbf{x}}} \quad \Rightarrow \quad \bar{\mathbf{y}}_x = Q_{xy}\left(\frac{\pi}{2}\right) * \bar{x} ;$$

$$\bar{z}_x = \mathcal{U}\left(\mathcal{V}\left(Q_{xy}\right)\right) = \frac{\mathcal{V}\left(Q_{xy}\right)}{\left|\mathcal{V}\left(Q_{xy}\right)\right|} .$$

Note that $\bar{\mathbf{y}}_{\mathrm{x}} \neq \bar{\mathbf{y}}$. It is a projection of $\bar{\mathbf{y}}$ into the plane perpendicular to $\bar{\mathbf{x}}$. The unit vector $\bar{\mathbf{z}}_x$ is not a projection of $\bar{\mathbf{z}}$ into the same plane, but the third orthogonal vector to complete the frame of reference. The definition of $\bar{\mathbf{z}}_x$ is not of great importance because it does not enter into the calculation of the rotation quaternion as long as it completes the frame of reference. The calculation of the rotation quaternion depends on only two axes. Note, however, that in as much as they are based on different vector pairs, one must use the **y** and **z** axes for computing the rotation quaternion for the $\bar{\mathbf{y}}$ frame of reference and the **z** and **x** vectors for computing the rotation quaternion for the $\bar{\mathbf{z}}$ frame of reference. When computing the frames for the transformed **y** and **z** axes, the ratio is $\mathbf{z}'$ to $\bar{\mathbf{y}}$ for the $\bar{\mathbf{y}}$ axis and $\mathbf{x}'$ to $\bar{\mathbf{z}}$ for the $\bar{\mathbf{z}}$ axis. Thus one maintains the same handedness for the frames. No amount of rotation will transform a right-handed coordinate system into a left-handed coordinate system.

The orientation shear is the ratio of the frames after the strain to the frame prior to the strain.

$$\boldsymbol{O}_x = \frac{\boldsymbol{f}_{\bar{x}}}{\boldsymbol{f}}\;;\quad \boldsymbol{O}_y = \frac{\boldsymbol{f}_{\bar{y}}}{\boldsymbol{f}}\;;\quad \boldsymbol{O}_z = \frac{\boldsymbol{f}_{\bar{z}}}{\boldsymbol{f}}\,.$$

It is possible to have three different quaternions for the transformation in orientation. Each is a transformation of the same starting frame.

$$\boldsymbol{F}_S = \begin{bmatrix} \boldsymbol{o}_{\bar{\mathbf{x}}} \\ \boldsymbol{o}_{\bar{\mathbf{y}}} \\ \boldsymbol{o}_{\bar{\mathbf{z}}} \end{bmatrix} * \boldsymbol{f} * \begin{bmatrix} \boldsymbol{o}_{\bar{\mathbf{x}}}^{-1} \\ \boldsymbol{o}_{\bar{\mathbf{y}}}^{-1} \\ \boldsymbol{o}_{\bar{\mathbf{z}}}^{-1} \end{bmatrix} = \boldsymbol{O}_S * \boldsymbol{f} * \boldsymbol{O}_S^{-1}$$

Since the axis of rotation cannot be aligned with all three frame axes and may not be aligned with any of them, it is necessary to write the transformation using the half angle formulation.

$$\boldsymbol{f}_w = \boldsymbol{o}_w * \boldsymbol{f} * \boldsymbol{o}_w^{-1}\;;\quad \angle \mathbf{o}_w = \frac{\angle \boldsymbol{O}_w}{2}\,,\quad \boldsymbol{w} = \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\,.$$

If the three orientation shears are the same quaternion, then there has been no distortion of the relationships between the coordinate axes. The frame may have been rotated, but it was not squashed. If the orientation shears have non-zero angles, then there was rotation. If they are scalars, then the strain was a translation or a change in scale.

Summary

These are fairly simple definitions of the three types of strain. Location strain is the $\Delta\boldsymbol{\lambda}$ term in the description of change of location, $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \Delta\boldsymbol{\lambda}$. Extension strain is the three quaternion array $\mathcal{E}_Q$ in the expression $\mathbf{V}' = \mathcal{E}_Q * \mathbf{V}$. Orientation strain is the three quaternion array $\boldsymbol{O}_S$ in the expression $\boldsymbol{F}_S = \boldsymbol{O}_S * \boldsymbol{f} * \boldsymbol{O}_S^{-1}$. The actual calculation of the strains may be somewhat complex and long, but the principles are straight-forward and intuitive.