Determining the Ellipse That Fits Three Vectors

The equations for an ellipse that may be tilted ϕ relative to the coordinates are written as follows.

$$x = a\,cos(\theta + \phi)\,,$$
$$y = b\,sin(\theta + \phi)\,.$$

We have three vectors that point to points on such an ellipse, but we do not know the values of a, b, or ϕ. However we can write three sets of equations of the following type with the values of x_n and $y_n$or each vector.

$$x_n = a\,cos(\theta_n + \phi)\,,$$
$$y_n = b\,sin(\theta_n + \phi),\ \text{where}$$
$$\theta_n = tan^{-1}\frac{y_n}{x_n}\,.$$

These equations can be written with the trigonometric functions expanded.

$$x_n = a\left[cos\theta_n\,cos\phi - sin\theta_n\,sin\phi\right],$$
$$y_n = b\left[sin\theta_n\,cos\phi + cos\theta_n\,sin\phi\right].$$

Solving for the cosine of the offset, ϕ, requires some algebraic manipulation.

$$\frac{x_n}{a\,sin\theta_n} = \frac{cos\theta_n}{sin\theta_n} * cos\phi - sin\phi\,,$$
$$\frac{y_n}{b\,cos\theta_n} = \frac{sin\theta_n}{cos\theta_n} * cos\phi + sin\phi\,.$$

The equations may be combined and solved for cos ϕ.

We start by remembering the Pythagorean theorem.

$$x_n^2 + y_n^2 = r_n^2 \quad \Leftrightarrow \quad r_n = \sqrt{x_n^2 + y_n^2}$$

Also, if we substitute the definition of θ into the expression then the equation may be reduced to a simpler expression in x and y.

$$sin\theta = \frac{y}{r}\ \text{and}\ cos\theta = \frac{x}{r}\,.$$

Combining the two equations and applying these definitions will give a formula for cos ϕ.

$$cos\phi\left[\frac{sin\theta_n}{cos\theta_n}+\frac{cos\theta_n}{sin\theta_n}\right]=\frac{x_n}{a\,sin\theta_n}+\frac{y_n}{b\,cos\theta_n}$$

$$cos\phi\left[\frac{x_n}{y_n}+\frac{y_n}{x_n}\right]=\frac{x_n r_n}{a y_n}+\frac{y_n r_n}{b x_n}$$

$$cos\phi\left[\frac{x_n^2+y_n^2}{x_n y_n}\right]=cos\phi\left[\frac{r_n^2}{x_n y_n}\right]=\frac{x_n r_n}{a y_n}+\frac{y_n r_n}{b x_n}$$

$$cos\phi=\left[\frac{x_n r_n}{a y_n}+\frac{y_n r_n}{b x_n}\right]*\left[\frac{x_n y_n}{r_n^2}\right]$$

$$=\left[\frac{x_n^2}{a}+\frac{y_n^2}{b}\right]*\frac{1}{r_n}$$

This expression may be written in a slightly different form by making some substitutions.

$$a'=\frac{1}{a};\quad b'=\frac{1}{b};\quad \alpha_n=\frac{x_n^2}{\sqrt{x_n^2+y_n^2}};\quad \beta_n=\frac{y_n^2}{\sqrt{x_n^2+y_n^2}}.$$

Finally, we can write out the equation for $cos\phi$ in a linear equation.

$$cos\phi=\alpha_n*a'+\beta_n*b',\quad n=1,2,3.$$

If we express this as a matrix equation, it would look as follows.

$$\begin{bmatrix}\alpha_1 & \beta_1 & -1\\ \alpha_2 & \beta_2 & -1\\ \alpha_3 & \beta_3 & -1\end{bmatrix}*\begin{bmatrix}\boldsymbol{a'}\\ \boldsymbol{b'}\\ cos\phi\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

We compute the values of the variables in the 3x3 matrix and then compute the values by solving the resulting matrix equation. The variables can then be computed by inverting the expressions in their definitions.

$$a=\frac{1}{a'};\quad b=\frac{1}{b'};\quad \phi=cos^{-1}(cos\phi).$$