Shear

At rest, description of an unstressed block

Imagine a block of material that is initially at rest, with no external forces. For that block we pick a convenient null point and an internal coordinate system. A convenient point is set to be the origin of the coordinate system. A set of basis vectors {**x, y, z**} is constructed at the origin, arranged so that they point in the three cardinal directions of the coordinate system.

Every point in the block is specified relative to the origin by a ***location*** vector, $\boldsymbol{\lambda}$, that extends from the origin to the point. The value of $\boldsymbol{\lambda}$ depends upon the choice of origin, but, if the origin is changed, one can easily compute the new values of $\boldsymbol{\lambda}$, by adding the location of the new origin in the old coordinate system to the location vector relative to the old origin.

$$\boldsymbol{\lambda}'_0 = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}'_{\text{Origin}}$$

A judicious choice of the origin can often greatly simplify an analysis.

At each point, an extension vector, $\boldsymbol{\varepsilon}$, is constructed that is the sum of three unit vectors aligned with the coordinate axes. The extension vectors can be viewed as test vectors, for visualizing the distortion that occurs in a region of the block.

An ***extension*** vector, $\boldsymbol{\varepsilon}$, may be resolved into coordinates by projecting it upon a set of three linearly independent vectors.

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \circ \mathbf{x}' + \boldsymbol{\varepsilon} \circ \mathbf{y}' + \boldsymbol{\varepsilon} \circ \mathbf{z}' = \boldsymbol{\varepsilon}_{\text{x}} + \boldsymbol{\varepsilon}_{\text{y}} + \boldsymbol{\varepsilon}_{\text{z}} \,,$$

where $\mathbf{x}'$, $\mathbf{y}'$, and $\mathbf{z}'$ are linearly independent.

If one selects a reference set of coordinates $\{\mathbf{x}', \mathbf{y}', \mathbf{z}'\}$ then $\boldsymbol{\varepsilon}$ is the diagonal of a box in that coordinate system. The volume of that box is given by the following expression, which is equivalent to the triple product of vector analysis.

$$\text{V} = \mathcal{S}\left[\mathcal{V}\left(\boldsymbol{\varepsilon}_{\mathbf{x}'} * \boldsymbol{\varepsilon}_{\mathbf{y}'}\right) * \boldsymbol{\varepsilon}_{\mathbf{z}'}\right] = \mathcal{S}\left[\mathcal{V}\left(\boldsymbol{\varepsilon}_{\mathbf{z}'} * \boldsymbol{\varepsilon}_{\mathbf{x}'}\right) * \boldsymbol{\varepsilon}_{\mathbf{y}'}\right] = \mathcal{S}\left[\mathcal{V}\left(\boldsymbol{\varepsilon}_{\mathbf{y}'} * \boldsymbol{\varepsilon}_{\mathbf{z}'}\right) * \boldsymbol{\varepsilon}_{\mathbf{x}'}\right]$$

Note that the volume of the box depends upon the choice of coordinates. It would be usual to choose the universal coordinates of the block as the box's sides, but other coordinates may be more revealing in some situations. The volume will always be zero if one of the coordinates is aligned with the extension vector and maximal when the coordinate axes are orthogonal and the projections most nearly equal.

At each location in the block, one selects a frame of reference, $\boldsymbol{f} = \{\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3}\}$, to express the ***orientation*** at that location. In an unstressed block, it is usually simplest to choose a frame that is aligned with the coordinate system.

$$\boldsymbol{f} = \left\{ \frac{\mathbf{x}}{|\mathbf{x}|}, \frac{\mathbf{y}}{|\mathbf{y}|}, \frac{\mathbf{z}}{|\mathbf{z}|} \right\}$$

However, a judicious choice of the orientation frames of reference may occasionally greatly simplify an analysis or highlight an effect. Orientation is arbitrary, but some choices generally seem more natural to the structure under study. For instance, the directions back to front, bottom to top, and left to right side are often the most intuitive choices of axes for a frame of reference.

Strain

Strain is the change in a material matrix when it is distorted by forces. The forces that produce the strain are called ***stress***. In this essay, we will be mostly concerned with the distortions and little concerned with the details of the stress forces. In particular, we are concerned with the mathematical description of strain. It will be argued that there are several types of strain and different types of strain may be relevant in different situations.

It is easy enough to describe the transformation associated with strain in particular instances, but we have to be sure to express the change in language that is as generally applicable as possible. Language that captures the change and nothing more than the change in a useful mathematical concept.

Location Strain

If the block is stressed, then we expect points within the block to shift in response to the points of application of the forces, the directions of the forces, and the strengths of the forces. As the material of the block shifts, the location vectors of points within the block change in an consistent manner.

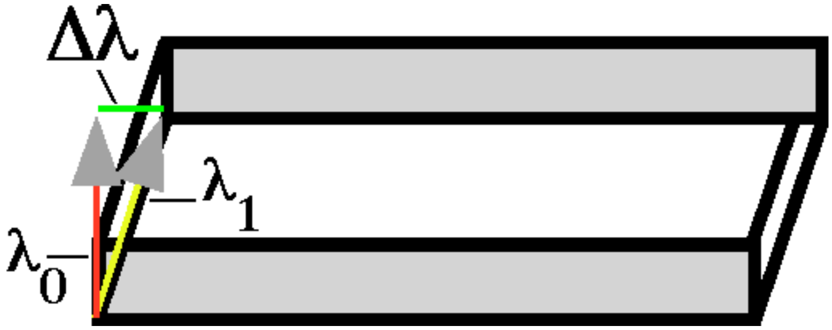
Let the original location of an internal point be indicated by the vector $\boldsymbol{\lambda}_0$ and its location after the forces are applied be $\boldsymbol{\lambda}_1$. The difference between the locations is $\boldsymbol{\Delta\lambda} = \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_0$. If location is one-dimensional and the displacement is uniform and proportional to location, then it would make sense to define the coefficient of strain as the ratio of the change to the location.

$$\sigma_\lambda = \frac{\lambda_1 - \lambda_0}{\lambda_0} = \frac{\Delta\lambda}{\lambda_0}$$

For a length of λ, the strain is $\Delta\lambda = \sigma_\lambda * \lambda$. The coefficient of strain is a function of the magnitude of the force producing the strain.

Sometimes we are interested in lengthening or contraction in one dimension and this definition is completely appropriate. It is normally how strain is defined, but it depends on selecting the axis of lengthening or contraction as the direction of the measurement. In principle, it is always possible to resolve all the forces acting at a point into a single force that has a single direction of action. Consequently, this approach is, in principle, sufficient. However, in a three-dimensional matrix the force is not necessarily in a constant direction at all points, so we will try to develop am more general formulation.

Location Strain in a Three-dimensional Matrix



In the example box illustrated above, we have chosen the box to line up with the direction of the force so the magnitudes of the vectors can be used. In this situation, the coefficient of strain is given by the following formula.

$$\boldsymbol{\sigma} = \frac{|\Delta\boldsymbol{\lambda}|}{|\boldsymbol{\lambda}_0|} * \frac{\mathbf{F}}{|\mathbf{F}|}.$$

The magnitude of the strain, $|\Delta\boldsymbol{\lambda}|$, is divided by the location relative to the origin, $|\lambda_0|$, the length in this case, and it is in the direction of the force, $\mathbf{F}$. Consequently, the coefficient of strain, $\boldsymbol{\sigma}$, is a vector.

If we look at this relation more closely, it is apparent that the relevant interval is not the length of the location vector. If we started with $\boldsymbol{\lambda}_1$, then the length would not be the magnitude of the $\boldsymbol{\lambda}_1$ vector, but the length of the $\boldsymbol{\lambda}_0$, vector, that is the projection of the location vector upon a unit vector perpendicular to the force vector. Let $\mathbf{f}$ be a unit vector in the direction of the force and let $\mathbf{p}$ be a unit vector perpendicular to $\mathbf{f}$ that points from the point of application of the force towards the origin of the coordinate system. If $\boldsymbol{\lambda}_A$ is the point of application of the force and $\boldsymbol{\lambda}_0$ is the origin for the location vectors, then there is a vector from the origin to the point of application, $\Lambda = \boldsymbol{\lambda}_A - \boldsymbol{\lambda}_0$, that lies in the same plane with the perpendicular to the force vector, so we can compute the quaternion that that rotates the force vector into $\boldsymbol{\Lambda}$. Both are reduced to unit vectors so that the rotation quaternion is a unit quaternion.

$$\boldsymbol{Q}_{f:\Lambda} = \frac{\boldsymbol{\Lambda}}{|\Lambda|\,\mathbf{f}}.$$

To obtain the perpendicular vector, $\mathbf{p}$, one rotates the unit vector in the direction of the force though 90° in the direction of $\boldsymbol{\Lambda}$.

$$\mathbf{p} = \boldsymbol{Q}_{\mathbf{f}:\Lambda}\left(\theta = \frac{\pi}{2}\right) * \mathbf{f}.$$

This allows us to write down the expression for the new location after a strain.

$$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \boldsymbol{\sigma}(\boldsymbol{\lambda}_0 \circ \mathbf{p}), \quad \boldsymbol{\sigma} = \frac{|\Delta\,\boldsymbol{\lambda}|}{|\boldsymbol{\lambda}_0|}\mathbf{f}.$$

Note that the second term in the expression is a scalar times a vector.

$(\boldsymbol{\lambda}_0 \circ \mathbf{p})$ is a scalar and $\boldsymbol{\sigma}$ is a vector .

In the case of uniform linear strain the strain is a vector, as just illustrated.

Extension Strain

If the block of material is visualized as having a regular array of "marker" points at regular intervals throughout the block, then extension may be defined relative to those markers and distortions of the material in the block may be expressed as changes in the array of markers. This interpretation leads to a definition of extension strain.

One could derive extension shear from location shear since extension is the difference between two locations. Therefore, it resembles location strain in being a ratio of two vectors, but the dependence on the origin drops out, because of one location being subtracted from the other. There may still be a dependence upon location, because the location strain may be a function of location.

Extension is the difference between two specific locations, $\boldsymbol{\lambda}_\alpha$ and $\boldsymbol{\lambda}_\beta$, which shift relative to each other during a strain.

$$\boldsymbol{\varepsilon}_0 = \boldsymbol{\lambda}_{\beta 0} - \boldsymbol{\lambda}_{\alpha 0}\,, \quad \boldsymbol{\varepsilon}_1 = \boldsymbol{\lambda}_{\beta 1} - \boldsymbol{\lambda}_{\alpha 1}$$

The difference between the two extensions is the ratio of the extension vectors.

$$\boldsymbol{\sigma}_\varepsilon = \frac{\boldsymbol{\varepsilon}_1}{\boldsymbol{\varepsilon}_0}$$

One extension that is often useful to examine is the diagonal of a small box that is a cube in the unstressed block and a rhomboid in the stressed block. There are also three anti-diagonals that extend from one of the proximal corners of the box to the opposite distal corner. Let us agree to name them for the proximal corner, so that the vector from the tip of the **z** unit vector to the sum of the **x** and **y** unit vectors is the **z** anti-diagonal. As mentioned above, the diagonal may be used to compute the volume of the box formed by the three coordinate axes. Changes in the volume of the box may indicate local compression or de-compression of the material of the block.

Orientation Strain

Each point in the block of material has an associated orientation. In the unstressed block it would be natural to let all the orientations be the same and the orientation axes to be aligned with the coordinate axes, however, there is no necessity to do so. Another arrangement might be more useful in some circumstances.

When the block is stressed, it will often lead to internal rotation of the substance of the block. When that happens, the orientation of the substance will change. Note that there are many situations in which the substance of the block will only translate. The illustration above shows a linear shear in which there is only translation. When there is only translation, then there is no change in orientation. There must be rotation to change orientation. That is the principal difference between linear strain and rotational strain.

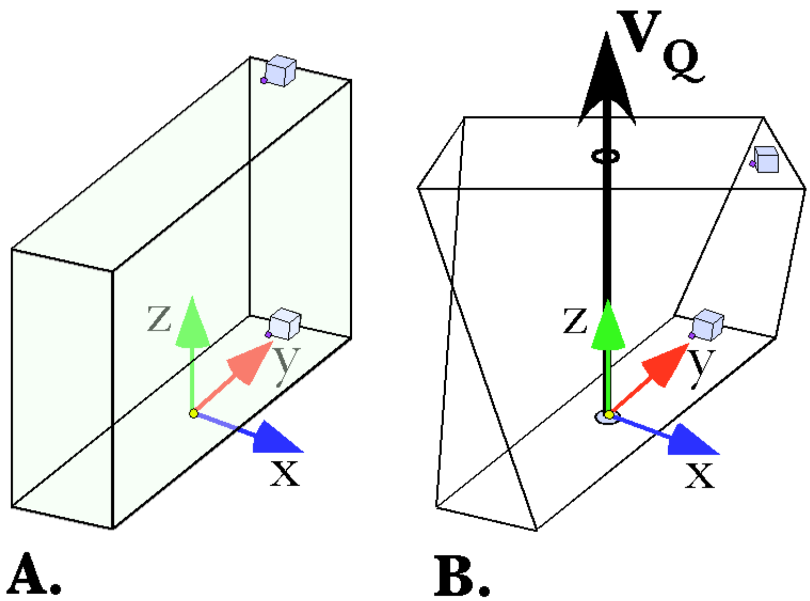
Orientation strain is defined similarly to the previous two types of strain. The change in orientation is the ratio of the orientations. Orientation is encoded in the frame of reference for the locations.

$$\boldsymbol{\sigma}_f = \frac{\boldsymbol{f}_1}{\boldsymbol{f}_0}$$

Orientation strain is also a quaternion, however, it is not as straight-forward to take the ratio of two orientations as it is two vectors. Basically, it is a matter of finding the rotation that carries the initial orientation into the final orientation. This has been described in detail elsewhere (Langer, 2004). In general terms, it involves finding the rotation, $\boldsymbol{Q}_{\mathrm{Sw}}$, that swings one axis in the initial frame of reference into alignment with its direction in the final frame of reference, then rotation the frame about that axis, $\boldsymbol{Q}_{\mathrm{Sp}}$, to bring the other axes into alignment. Both rotations are quaternions. The product of those rotation quaternions, in the opposite order, $\boldsymbol{Q}_{\mathrm{Sp}} * \boldsymbol{Q}_{\mathrm{Sw}}$, is the quaternion that rotates the initial orientation into the final orientation. However, since the combined quaternion is a conical rotation and the two component rotations are planar rotations, suitable adjustments must be made to the angles of the quaternions, namely halving their angles before multiplying them.

$$\boldsymbol{f}_1 = \boldsymbol{q}_{Sp} * \boldsymbol{q}_{Sw} * \boldsymbol{f}_0 * \boldsymbol{q}_{Sw}^{-1} * \boldsymbol{q}_{Sp}^{-1}; \text{ where } \angle \boldsymbol{q} = \frac{\angle \boldsymbol{Q}}{2}.$$

Once we establish that there is always a quaternion that will rotate the initial frame of reference into the final frame of reference, then we know that there is always an orientation strain that describes the transformation.


z
y
x
z
y
x
V_Q
A.
B.