Torsional Strain

In torsion the matrix rotates about a center of rotation. In the framework that will be used here, one plate rotates relative to the other plate. The matrix flows in such a manner as to minimize the change of volume everywhere in the matrix. In this situation, the shear is proportional to the distance from the axis of rotation and the magnitude of the angular excursion. It is also a function of the distance of the location from the fixed plate. In fact, the situation is comparable to the arrangement for linear shear, except that the natural reference framework is cylindrical rings concentric about the axis of rotation.

If the location prior to the torsion is $\boldsymbol{\lambda}_0$ and after is $\boldsymbol{\lambda}_1$, then $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \Delta\boldsymbol{\lambda}$. For now, it will be assumed that the axis of rotation, $\boldsymbol{\rho}$, is perpendicular to the two plates, so that the perpendicular is a unit vector parallel with the axis of rotation.

$$\mathbf{p} = \frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|}.$$

The initial location vector is resolved into two component vectors, one parallel to the perpendicular, **h**, and one perpendicular to it, **r**.

$$\mathbf{h} = \boldsymbol{\lambda}_0 \bullet \mathbf{p} \quad \text{and} \quad \mathbf{r} = \boldsymbol{\lambda}_0 - \mathbf{h}.$$

We will also need to designate a tangential unit vector, $\boldsymbol{\tau}$, that is perpendicular to $\mathbf{r}$ and $\mathbf{h}$.

$$\boldsymbol{\tau} = \frac{\frac{\boldsymbol{\rho}}{\mathbf{r}}}{\left|\frac{\boldsymbol{\rho}}{\mathbf{r}}\right|}.$$

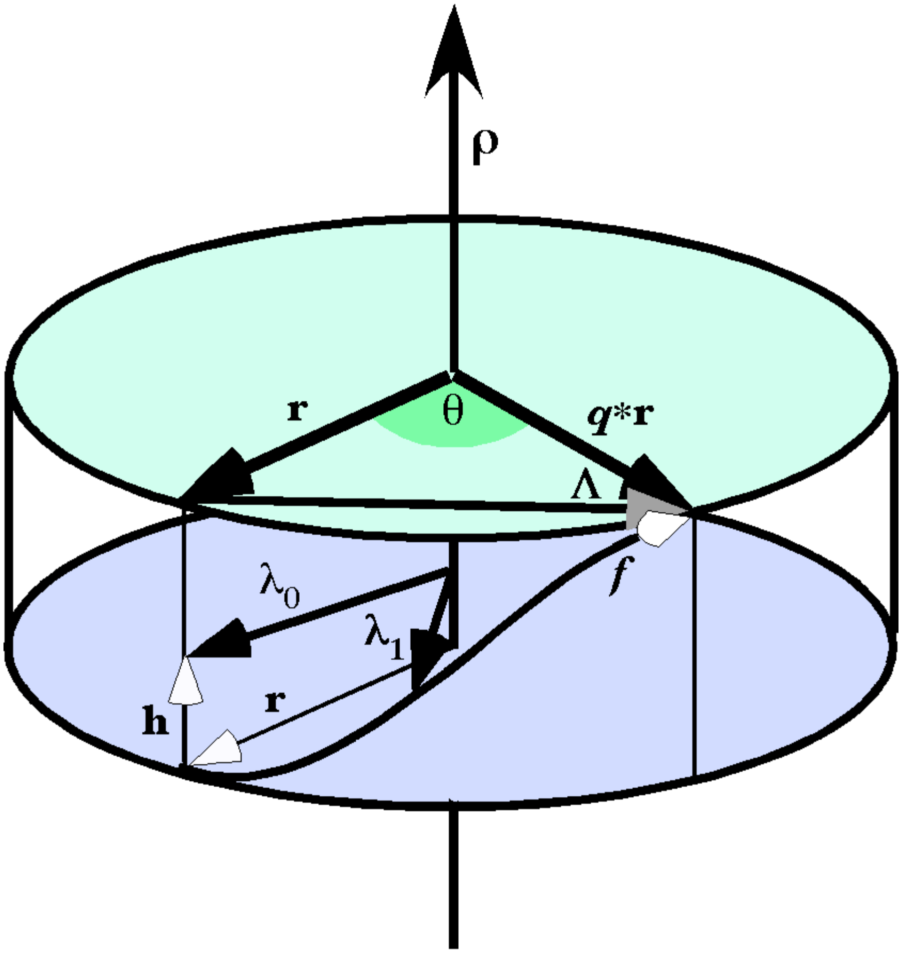
The total shear, the movement at the moving plate is the angular excursion times the radial distance, concentric with the axis of rotation.

$$|\boldsymbol{\sigma}| = \theta * |\mathbf{r}|$$

However, the shear is along a circular trajectory, therefore it is necessary to compute the final location and subtract the initial location to obtain the actual excursion.

$$\boldsymbol{\Lambda}(\boldsymbol{\lambda}_0) = \boldsymbol{q} * \mathbf{r}(\boldsymbol{\lambda}_0) - \mathbf{r}(\boldsymbol{\lambda})_0, \quad \boldsymbol{q} = cos\theta + sin\theta * \boldsymbol{\rho}.$$

Since $|\boldsymbol{\Lambda}| \leq |\boldsymbol{\sigma}|$, there is a tendency for the matrix to be drawn centrally, but that space is already occupied by the material of the more central rings, so the matrix must flow in its own ring. Still, there will be in centrally directed force that will tend to compress the matrix. Since a ring is essentially a flat surface we can, as it were, peel it away to treat the flow as we did in linear shear and lay it back into its original position to see how that flow, ***f***, appears *in situ*.



The upper (green) plate is rotated through an angular excursion of θ **about an axis of rotation,** ρ**. Producing a rotational shear between the two plates**. For locations between the plates, $\boldsymbol{\lambda}_0$, the flow of the matrix is in a ring concentric with the axis of rotation to an extent that depends upon the distance from the location to the fixed plate (blue) and the separation between the two plates, 2χ.

We have already derived the expression for $\Delta\boldsymbol{\lambda}$, given $\boldsymbol{\lambda}_0$, when the shear is linear.

$$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \Delta\boldsymbol{\lambda},$$

$$\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 + \frac{3}{4}\frac{\sigma}{\chi}\left(\frac{x^3}{3} - x^2\chi\right),$$

$$x = \boldsymbol{\lambda}_0 \bullet \mathbf{p}.$$

With several changes, a similar expression will describe torsional strain. The total shear is an angular excursion, $\varsigma = \theta * |\mathbf{r}|$, and the perpendicular offset is $x = |\mathbf{h}|$. The $\boldsymbol{\Delta \lambda}$ is also expressed as an angular excursion.

$$\phi = \frac{|\boldsymbol{\Delta \lambda}|}{|\mathbf{r}|}.$$

The distance $\boldsymbol{\Delta \lambda}$ is the distance within the ring, rather than the chord that connects the initial and final locations. Consequently, the expression relating the initial and final locations can be written as follows.

$$\boldsymbol{\lambda}_1 = \boldsymbol{T} * \boldsymbol{\lambda}_0 ,$$
$$\boldsymbol{T} = cos\,\phi + sin\,\phi * \boldsymbol{\rho} ,$$
$$\phi = \frac{3}{4}\frac{\varsigma}{\chi}\left(\frac{x^3}{3} - x^2\chi\right) * \frac{1}{|\mathbf{r}|} ,$$
$$x = |\mathbf{h}| = \boldsymbol{\lambda}_0 \bullet \mathbf{p} = \boldsymbol{\lambda}_0 \bullet \frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|} .$$

With torsional shear, the final location is more concisely expressed as a quaternion product with the initial location. Note that the angular excursion can be reduced to a scalar factor times the total angular shear.

$$\phi = \theta * \frac{3}{4\chi}\left[\frac{x^3}{3} - x^2\chi\right]$$

This means that the angular excursion of the flow is independent of the distance from the axis of rotation, which is in accord with our expectations. This is of physical importance, because it means that the strain is uniform throughout the radius of the matrix.

There is a type of torsion in which the strain is not uniform radially. If the torque force is applied to the outer surface of the matrix, as in unscrewing a jar lid, then the strain differential is between the outer surface and the axis of rotation. If the tangential force is assumed to be uniform over the outer surface, then the strain is greatest at the outer surface and it is attenuated as one moves centrally. However, unless the center is fixed, the mass will begin to rotate as a unit and the strain will be resolved. A variant of this scenario that is potentially more interesting is when the mass is fixed or retarded by other forces and a torque force is applied to a portion of the outer surface. For instance, when the mass is a bone shaft and the torque is due to the pull at a muscle attachment. That problem is far more difficult and therefore will be deferred until we have more experience with strain.

Extension Matrix

In the following, we will consider the torsional strain described above. As with linear shear, the local strain is most dependent upon the distance from the fixed plate, 'x'. If we shift the initial location centrally or peripherally ($\mathbf{\Delta r}$), there is no change in the angular excursion of the location, $\boldsymbol{\Delta\lambda}_\theta$. There is a change in the actual distance moved, but it is proportional to $|\mathbf{r}|$. If the shift is around the ring ($\Delta\theta$), then there is no change in the magnitude of the strain excursion. Obviously, the strain is in a different direction, because we have moved upon a circle and the strain is tangential to the circle. A shift in the location relative to the fixed plate ($|\mathbf{h}|$), will cause a change in the magnitude of the strain, just as it did in linear strain.

The description of the strain was given above. It is repeated here so that we can examine it for its dependencies upon the three types of variation of location. The dependencies may be in the initial location or the strain quaternion.

$$\boldsymbol{\lambda}_1 = \boldsymbol{T} * \boldsymbol{\lambda}_0(\mathrm{r},\theta,\mathrm{x}),$$

$$\boldsymbol{T} = cos\,\phi + sin\,\phi * \boldsymbol{\rho},$$

$$\phi = \frac{3}{4}\frac{\varsigma}{\chi}\left[\mathrm{x} - \frac{1}{\chi^2}\left(\frac{\mathrm{x}^3}{3} - \frac{2\mathrm{x}^2\chi}{2} + \mathrm{x}\chi^2\right)\right] * \frac{1}{|\mathbf{r}|},$$

$$\mathrm{x} = |\mathbf{h}| = \boldsymbol{\lambda}_0 \bullet \mathbf{p} = \boldsymbol{\lambda}_0 \bullet \frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|},$$

$$\mathrm{r} = |\mathbf{r}|, \quad \varsigma = |\mathbf{r}| * \varphi.$$

If the radial distance of the initial location is changed, then the change in the final location is calculated as follows.

$$\boldsymbol{\lambda}_1(\mathrm{r}-\varepsilon) = [cos\,\kappa + sin\,\kappa\,\mathbf{k}] * \left[((\mathbf{r}-\boldsymbol{\varepsilon})\bullet\mathbf{i})\mathbf{i} + ((\mathbf{r}-\boldsymbol{\varepsilon})\bullet\mathbf{j})\mathbf{j} + ((\boldsymbol{r}-\boldsymbol{\varepsilon})\bullet\mathbf{k})\mathbf{k}\right],$$

$$\boldsymbol{\lambda}_1(\mathrm{r}+\varepsilon) = [cos\,\kappa + sin\,\kappa\,\mathbf{k}] * \left[((\mathbf{r}+\boldsymbol{\varepsilon})\bullet\mathbf{i})\mathbf{i} + ((\mathbf{r}+\boldsymbol{\varepsilon})\bullet\mathbf{j})\mathbf{j} + ((\boldsymbol{r}+\boldsymbol{\varepsilon})\bullet\mathbf{k})\mathbf{k}\right],$$

$$\kappa = \frac{3}{4}\frac{\varsigma}{\chi}\left[\mathrm{x} - \frac{1}{\chi^2}\left(\frac{\mathrm{x}^3}{3} - \mathrm{x}^2\chi + \mathrm{x}\chi^2\right)\right] * \frac{1}{|\mathbf{r}|}.$$

Let us rewrite the equations to simplify the mechanics of the calculation by concealing the unnecessary detail until it is needed.

$$\boldsymbol{\lambda}_1(\mathrm{r}-\varepsilon)=[\alpha+\beta\mathbf{k}]*\left[(\mathrm{r_i}-\varepsilon_\mathrm{i})\mathbf{i}+(\mathrm{r_j}-\varepsilon_\mathrm{j})\mathbf{j}+(\mathrm{r_k}-\varepsilon_\mathrm{k})\mathbf{k}\right],$$
$$\boldsymbol{\lambda}_1(\mathrm{r}+\varepsilon)=[\alpha+\beta\mathbf{k}]*\left[(\mathrm{r_i}+\varepsilon_\mathrm{i})\mathbf{i}+(\mathrm{r_j}+\varepsilon_\mathrm{j})\mathbf{j}+(\mathrm{r_k}+\varepsilon_\mathrm{k})\mathbf{k}\right].$$

The difference between these two locations is given by the following expression.

$$2\left[(\alpha\varepsilon_\mathrm{i}-\beta\varepsilon_\mathrm{j})\mathbf{i}+(\alpha\varepsilon_\mathrm{j}+\beta\varepsilon_\mathrm{i})\mathbf{j}\right]$$

The first extension matrix component is the ratio of this difference to twice the offset.

$$\boldsymbol{Q}_\mathrm{r}=\frac{2\left[(\alpha\varepsilon_\mathrm{i}-\beta\varepsilon_\mathrm{j})\mathbf{i}+(\alpha\varepsilon_\mathrm{j}+\beta\varepsilon_\mathrm{i})\mathbf{j}\right]}{2(\varepsilon_\mathrm{i}\,\mathbf{i}+\varepsilon_\mathrm{j}\,\mathbf{j})}=\frac{(\alpha\varepsilon_\mathrm{i}-\beta\varepsilon_\mathrm{j})\mathbf{i}+(\alpha\varepsilon_\mathrm{j}+\beta\varepsilon_\mathrm{i})\mathbf{j}}{\varepsilon_\mathrm{i}\,\mathbf{i}+\varepsilon_\mathrm{j}\,\mathbf{j}}.$$

We do not change the fundamental geometrical relationships if we rotate our coordinate system so that the **i** axis is aligned with the **r** vector. However, doing so makes it apparent that the extension quaternion is the rotation quaternion for the rotation shear. The radial offset is rotated just like the initial location.

$$\boldsymbol{Q}_\mathrm{r}=\frac{\alpha\varepsilon\mathbf{i}+\beta\varepsilon\mathbf{j}}{\varepsilon\mathbf{i}}=\frac{\alpha\,\mathbf{i}+\beta\mathbf{j}}{\mathbf{i}}=(\alpha\,\mathbf{i}+\beta\mathbf{j})*-\mathbf{i}=\alpha+\beta\mathbf{k}=\boldsymbol{T}.$$

Therefore, the final position for a location plus a radial offset is the sum of the transformations of the two components.

$$\boldsymbol{\lambda}_1=\boldsymbol{T}*(\boldsymbol{\lambda}_0+\varepsilon_\mathrm{r})=\boldsymbol{T}*\boldsymbol{\lambda}_0+\boldsymbol{T}*\varepsilon_\mathrm{r}.$$

There is no explicit dependence of the extension matrix upon θ, other than the initial location's component and that was embedded in the analysis immediately above. Therefore the second component of the expansion matrix is unity.

$$\boldsymbol{\lambda}_1=\boldsymbol{T}*(\boldsymbol{\lambda}_0+\varepsilon_\theta)=\boldsymbol{T}*\boldsymbol{\lambda}_0+\varepsilon_\theta,\ \text{thus}$$
$$\boldsymbol{Q}_\theta=\frac{\varepsilon_\theta}{\varepsilon_\theta}=1.0.$$

The vertical component is more complex than the first two components. The is a complex dependence upon the $|\mathbf{h}|=\mathrm{x}$ variable, but it is much like the vertical component for linear shear. We start much as for the radial component, except κ is not a constant.

$$\boldsymbol{\lambda}_1(x-\varepsilon)=\left[cos\frac{\breve{\kappa}}{2}+sin\frac{\breve{\kappa}}{2}\mathbf{k}\right]*\left[r_i\mathbf{i}+r_j\mathbf{j}+(r_k-\varepsilon)\mathbf{k}\right]*\left[cos\frac{\breve{\kappa}}{2}-sin\frac{\breve{\kappa}}{2}\mathbf{k}\right],$$

$$\boldsymbol{\lambda}_1(x+\varepsilon)=\left[cos\frac{\widehat{\kappa}}{2}+sin\frac{\widehat{\kappa}}{2}\mathbf{k}\right]*\left[r_i\mathbf{i}+r_j\mathbf{j}+(r_k+\varepsilon)\mathbf{k}\right]*\left[cos\frac{\widehat{\kappa}}{2}-sin\frac{\widehat{\kappa}}{2}\mathbf{k}\right],$$

$$\breve{\kappa}=\frac{3}{4}\frac{\varsigma}{\chi^3}\left[-\frac{(x-\varepsilon)^3}{3}+(x-\varepsilon)^2\chi\right]*\frac{1}{|\mathbf{r}|},$$

$$\widehat{\kappa}=\frac{3}{4}\frac{\varsigma}{\chi^3}\left[-\frac{(x+\varepsilon)^3}{3}+(x+\varepsilon)^2\chi\right]*\frac{1}{|\mathbf{r}|}.$$

These equations reduce to equations that vary in the **i** and **j** terms.

$$\boldsymbol{\lambda}_1(x-\varepsilon)=\left(r_i\,cos\,\breve{\kappa}-r_j\,sin\,\breve{\kappa}\right)\mathbf{i}+\left(r_j\,cos\,\breve{\kappa}+r_i\,sin\,\breve{\kappa}\right)\mathbf{j}+(r_k-\varepsilon)\mathbf{k},$$

$$\boldsymbol{\lambda}_1(x+\varepsilon)=\left(r_i\,cos\,\widehat{\kappa}-r_j\,sin\,\widehat{\kappa}\right)\mathbf{i}+\left(r_j\,cos\,\widehat{\kappa}+r_i\,sin\,\widehat{\kappa}\right)\mathbf{j}+(r_k+\varepsilon)\mathbf{k}.$$

The difference between the two location transformation is obtained readily.

$$\Delta\boldsymbol{\lambda}=\boldsymbol{\lambda}_1(x+\varepsilon)-\boldsymbol{\lambda}_1(x-\varepsilon),$$

$$=\left[\left(r_i\,cos\,\widehat{\kappa}-r_j\,sin\,\widehat{\kappa}\right)-\left(r_i\,cos\,\breve{\kappa}-r_j\,sin\,\breve{\kappa}\right)\right]\mathbf{i}+\left[\left(r_j\,cos\,\widehat{\kappa}+r_i\,sin\,\widehat{\kappa}\right)-\left(r_j\,cos\,\breve{\kappa}+r_i\,sin\,\breve{\kappa}\right)\right]\mathbf{j}+2\varepsilon\mathbf{k},$$

$$=\left[r_i\left(cos\,\widehat{\kappa}-cos\,\breve{\kappa}\right)-r_j\left(sin\,\widehat{\kappa}-sin\,\breve{\kappa}\right)\right]\mathbf{i}+\left[r_j\left(cos\,\widehat{\kappa}-cos\,\breve{\kappa}\right)+r_i\left(sin\,\widehat{\kappa}-sin\,\breve{\kappa}\right)\right]\mathbf{j}+2\varepsilon\mathbf{k},$$

$$=-2*\left\{\left[r_i\,S-r_j\,C\right]\mathbf{i}-\left[r_j\,S+r_i\,C\right]\right\}\mathbf{j}+2\varepsilon\mathbf{k}\text{ where}$$

$$S=\left(sin\frac{\widehat{\kappa}+\breve{\kappa}}{2}*sin\frac{\widehat{\kappa}-\breve{\kappa}}{2}\right)\text{ and }C=\left(cos\frac{\widehat{\kappa}+\breve{\kappa}}{2}*sin\frac{\widehat{\kappa}-\breve{\kappa}}{2}\right).$$

The dependence upon the offset lies in the κ terms.

$$\frac{\widehat{\kappa}+\breve{\kappa}}{2}=\frac{1}{2}*\frac{3}{4}\frac{\varsigma}{\chi^3}\left[\frac{(x-\varepsilon)^3+(x+\varepsilon)^3}{3}+\chi\left[(x+\varepsilon)^2+(x-\varepsilon)^2\right]\right]*\frac{1}{|\mathbf{r}|}$$

$$=\frac{3}{4}\frac{\varsigma}{\chi^3}\left[\frac{x^3}{3}-x\varepsilon^2+x^2\chi+\varepsilon^2\chi\right]*\frac{1}{|\mathbf{r}|},\text{ and}$$

$$\frac{\widehat{\kappa}-\breve{\kappa}}{2}=\frac{1}{2}*\frac{3}{4}\frac{\varsigma}{\chi^3}\left[\frac{(x-\varepsilon)^3-(x+\varepsilon)^3}{3}+\chi\left[(x+\varepsilon)^2-(x-\varepsilon)^2\right]\right]*\frac{1}{|\mathbf{r}|}$$

$$=\frac{3}{4}\frac{\varsigma}{\chi^3}\left[-x^2\varepsilon-\frac{\varepsilon^3}{3}+2x\varepsilon\chi\right]*\frac{1}{|\mathbf{r}|}.$$

For computation, the third line of the expression for $\Delta\lambda$ is perfectly adequate. The quaternion that transforms a vertical off set of $\pm\varepsilon$ into $\Delta\lambda$ is the ratio of the second variable to the first .

$$\boldsymbol{Q}_{\mathbf{k}} = \frac{\left[\mathrm{r}_{\mathbf{i}}\left(cos\,\widehat{\kappa} - cos\,\breve{\kappa}\right) - \mathrm{r}_{\mathbf{j}}\left(sin\,\widehat{\kappa} - sin\,\breve{\kappa}\right)\right]\mathbf{i} + \left[\mathrm{r}_{\mathbf{j}}\left(cos\,\widehat{\kappa} - cos\,\breve{\kappa}\right) + \mathrm{r}_{\mathbf{i}}\left(sin\,\widehat{\kappa} - sin\,\breve{\kappa}\right)\right]\mathbf{j} + 2\varepsilon\mathbf{k}}{2\varepsilon\mathbf{k}}$$

$$= \left[\left[\mathrm{r}_{\mathbf{i}}\left(cos\,\widehat{\kappa} - cos\,\breve{\kappa}\right) - \mathrm{r}_{\mathbf{j}}\left(sin\,\widehat{\kappa} - sin\,\breve{\kappa}\right)\right]\mathbf{i} + \left[\mathrm{r}_{\mathbf{j}}\left(cos\,\widehat{\kappa} - cos\,\breve{\kappa}\right) + \mathrm{r}_{\mathbf{i}}\left(sin\,\widehat{\kappa} - sin\,\breve{\kappa}\right)\right]\mathbf{j} + 2\varepsilon\mathbf{k}\right] * \frac{-2\varepsilon\mathbf{k}}{4\varepsilon^2}$$

$$= 1 - \frac{\left[\mathrm{r}_{\mathbf{j}}\left(cos\,\widehat{\kappa} - cos\,\breve{\kappa}\right) + \mathrm{r}_{\mathbf{i}}\left(sin\,\widehat{\kappa} - sin\,\breve{\kappa}\right)\right]}{2\varepsilon}\mathbf{i} + \frac{\left[\mathrm{r}_{\mathbf{i}}\left(cos\,\widehat{\kappa} - cos\,\breve{\kappa}\right) - \mathrm{r}_{\mathbf{j}}\left(sin\,\widehat{\kappa} - sin\,\breve{\kappa}\right)\right]}{2\varepsilon}\mathbf{j}$$

The extension matrix for torsion is the combination of the three quaternions that have just been computed.

$$\mathcal{E}_{\text{Torsion}} = \left\{\boldsymbol{Q}_{\mathbf{r}}, \boldsymbol{Q}_{\theta}, \boldsymbol{Q}_{\mathbf{k}}\right\}$$

$$= \left\{\boldsymbol{T}, 1.0, 1 - \frac{\left[\mathrm{r}_{\mathbf{j}}\left(cos\,\widehat{\kappa} - cos\,\breve{\kappa}\right) + \mathrm{r}_{\mathbf{i}}\left(sin\,\widehat{\kappa} - sin\,\breve{\kappa}\right)\right]}{2\varepsilon}\mathbf{i} + \frac{\left[\mathrm{r}_{\mathbf{i}}\left(cos\,\widehat{\kappa} - cos\,\breve{\kappa}\right) - \mathrm{r}_{\mathbf{j}}\left(sin\,\widehat{\kappa} - sin\,\breve{\kappa}\right)\right]}{2\varepsilon}\mathbf{j}\right\}.$$

Orientation Matrix

The net effect of a torsional strain is to carry a location around a concentric ring., Consequently, we would expect there to be a rotation of the frame of reference. To start, let us review the transformation of location.

$$\boldsymbol{\lambda}_1 = \boldsymbol{T} * \boldsymbol{\lambda}_0\left(\mathrm{r}, \theta, \mathrm{x}\right),$$
$$\boldsymbol{T} = cos\,\phi + sin\,\phi * \boldsymbol{\rho},$$
$$\phi = \frac{3}{4}\frac{\varsigma}{\chi}\left[\mathrm{x} - \frac{1}{\chi^2}\left(\frac{\mathrm{x}^3}{3} - \frac{2\mathrm{x}^2\chi}{2} + \mathrm{x}\,\chi^2\right)\right] * \frac{1}{|\mathbf{r}|}$$

If we add a small offset in the horizontal plane to the original location, then the new location is given by the following formula.

$$\boldsymbol{\lambda}_1\left(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}_{\mathbf{xy}}\right) = \boldsymbol{T} * \left(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}\right), \text{ where } \boldsymbol{T} = cos\,\phi + sin\,\phi * \boldsymbol{\rho},$$

$$\boldsymbol{\lambda}_1\left(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}_{\mathbf{xy}}\right) = \boldsymbol{T} * \left(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}\right) = \left(cos\frac{\phi}{2} + sin\frac{\phi}{2} * \mathbf{k}\right) * \left(\left(\mathrm{r}_{\mathbf{i}} + \varepsilon_{\mathbf{i}}\right)\mathbf{i} + \left(\mathrm{r}_{\mathbf{j}} + \varepsilon_{\mathbf{j}}\right)\mathbf{j} + \mathrm{r}_{\mathbf{k}}\mathbf{k}\right) * \left(cos\frac{\phi}{2} - sin\frac{\phi}{2} * \mathbf{k}\right)$$

$$= \left(\left(\mathrm{r}_{\mathbf{i}} + \varepsilon_{\mathbf{i}}\right)cos\,\phi - \left(\mathrm{r}_{\mathbf{j}} + \varepsilon_{\mathbf{j}}\right)sin\,\phi\right)\mathbf{i} + \left(\left(\mathrm{r}_{\mathbf{j}} + \varepsilon_{\mathbf{j}}\right)cos\,\phi + \left(\mathrm{r}_{\mathbf{i}} + \varepsilon_{\mathbf{i}}\right)sin\,\phi\right)\mathbf{j} + \mathrm{r}_{\mathbf{k}}\,\mathbf{k}$$

The transformation quaternion for horizontal offsets is this is the difference between the two $\boldsymbol{\lambda}_1$ vectors, divided by the offset.

$$
\begin{aligned}
\boldsymbol{Q}_H &= \frac{\boldsymbol{\lambda}_1(\boldsymbol{\lambda}_0 + \boldsymbol{\varepsilon}) - \boldsymbol{\lambda}_1(\boldsymbol{\lambda}_0)}{\boldsymbol{\varepsilon}} \\
&= \frac{\left(\varepsilon_i \cos\phi - \varepsilon_j \sin\phi\right)\mathbf{i} + \left(\varepsilon_j \cos\phi + \varepsilon_i \sin\phi\right)\mathbf{j}}{\varepsilon_i\,\mathbf{i} + \varepsilon_j\,\mathbf{j}} \\
&= \left[\left(\varepsilon_i \cos\phi - \varepsilon_j \sin\phi\right)\mathbf{i} + \left(\varepsilon_j \cos\phi + \varepsilon_i \sin\phi\right)\mathbf{j}\right] * \frac{-\left(\varepsilon_i\,\mathbf{i} + \varepsilon_j\,\mathbf{j}\right)}{\varepsilon_i^2 + \varepsilon_j^2} \\
&= \cos\phi + \sin\phi\,\mathbf{k} = \boldsymbol{T}
\end{aligned}
$$

The quaternion is the same rotation as was applied to the initial location so the first component of the orientation matrix is ***T***. $\boldsymbol{Q}_x = \boldsymbol{T}$. The transformed y axis is the initial **y** axis rotated as specified by the rotation quaternion ***T***.

As before, the change with vertical offsets is more complex. There is a shear component and a rotation component.

$$
\boldsymbol{\lambda}_1(x + \varepsilon) = \left[\cos\frac{\hat{\kappa}}{2} + \sin\frac{\hat{\kappa}}{2}\mathbf{k}\right] * \left[r_i\mathbf{i} + r_j\mathbf{j} + (r_k + \varepsilon_k)\mathbf{k}\right] * \left[\cos\frac{\hat{\kappa}}{2} - \sin\frac{\hat{\kappa}}{2}\mathbf{k}\right],
$$

$$
\hat{\kappa} = \frac{3}{4}\frac{\varsigma}{\chi^3}\left[-\frac{(x+\varepsilon)^3}{3} + (x+\varepsilon)^2\chi\right] * \frac{1}{|\mathbf{r}|},
$$

$$
\boldsymbol{\lambda}_1(x + \varepsilon) = \left(r_i \cos\hat{\kappa} - r_j \sin\hat{\kappa}\right)\mathbf{i} + \left(r_j \cos\hat{\kappa} + r_i \sin\hat{\kappa}\right)\mathbf{j} + \left(r_k + \varepsilon\right)\mathbf{k},
$$

$$
\boldsymbol{\lambda}_1(x) = \left(r_i \cos\kappa - r_j \sin\kappa\right)\mathbf{i} + \left(r_j \cos\kappa + r_i \sin\kappa\right)\mathbf{j} + \left(r_k\right)\mathbf{k},
$$

$$
\kappa = \frac{3}{4}\frac{\varsigma}{\chi^3}\left[-\frac{x^3}{3} + x^2\chi\right] * \frac{1}{|\mathbf{r}|},
$$

$$
\boldsymbol{\lambda}_1(x + \varepsilon) - \boldsymbol{\lambda}_1(x) = \left(r_i\left(\cos\hat{\kappa} - \cos\kappa\right) - r_j\left(\sin\hat{\kappa} - \sin\kappa\right)\right)\mathbf{i} + \left(r_j\left(\cos\hat{\kappa} - \cos\kappa\right) + r_i\left(\sin\hat{\kappa} - \sin\kappa\right)\right)\mathbf{j} + \varepsilon\mathbf{k},
$$

$$
= -2 * \left\{\left[r_i S - r_j C\right]\mathbf{i} - \left[r_j S + r_i C\right]\right\}\mathbf{j} + \varepsilon\mathbf{k}, \text{ where}
$$

$$
S = \left(\sin\frac{\hat{\kappa} + \kappa}{2} * \sin\frac{\hat{\kappa} - \kappa}{2}\right) \text{ and } C = \left(\cos\frac{\hat{\kappa} + \kappa}{2} * \sin\frac{\hat{\kappa} - \kappa}{2}\right)..
$$

The transformed **z** axis is given by the last equation. When it is all written out in terms of the parameters of the torsion, it is a complex expression. However, it is fundamentally a vector, so we can re-write it as follows.

$$
\boldsymbol{\Delta\lambda} = \hat{\alpha}\,\mathbf{i} + \hat{\beta}\,\mathbf{j} + \hat{\gamma}\,\mathbf{k}.
$$

The basis vectors of the transformed frame of reference can be written as in terms of the expressions just derived.

$$f_T = \left\{ \boldsymbol{T} * \mathbf{i}, \boldsymbol{T} * \mathbf{j}, \hat{\alpha}\,\mathbf{i} + \hat{\beta}\,\mathbf{j} + \hat{\gamma}\,\mathbf{k} \right\}.$$

Note that we can write the first two terms as simple quaternion multiplication because it is set that **i** and **j** lie in the plane of ***T***. If that were not true, then they would be written as conical rotations.

Since the transformation of the first two components rotates both identically, we can write the first component of the orientation matrix as the common transformation, that is, ***T***.

$$\boldsymbol{Q}_{\mathbf{x}\to\bar{\mathbf{x}}} = \frac{\boldsymbol{T} * \mathbf{i}}{\mathbf{i}} = \boldsymbol{T} * \mathbf{i} * -\mathbf{i} = \boldsymbol{T} = \cos\phi + \sin\phi * \mathbf{k}.$$

The second component depends upon finding the ratio of the third axis to the second axis. Then, the second axis is rotated through a right angle about the axis of the ratio to obtain its projection into the plane perpendicular to the second axis.

$$\boldsymbol{Q}_{\mathbf{yz}} = \frac{\hat{\alpha}\,\mathbf{i} + \hat{\beta}\,\mathbf{j} + \hat{\gamma}\,\mathbf{k}}{\boldsymbol{T} * \mathbf{j}} = \frac{1}{|\boldsymbol{T}|^2} * \hat{\alpha}\,\mathbf{i} + \hat{\beta}\,\mathbf{j} + \hat{\gamma}\,\mathbf{k} * \boldsymbol{T} * -\mathbf{j}.$$

The basis vectors are unit vectors and the rotation quaternion, T, is a unit quaternion, therefore, the expression reduces to a simpler form.

$$\begin{aligned}\boldsymbol{Q}_{\mathbf{yz}} &= \left(\hat{\alpha}\,\mathbf{i} + \hat{\beta}\,\mathbf{j} + \hat{\gamma}\,\mathbf{k}\right) * \left(\cos\phi + \sin\phi * \mathbf{k}\right) * -\mathbf{j}, \\ &= \left(\hat{\beta}\cos\phi - \hat{\alpha}\sin\phi\right) + \hat{\gamma}\cos\phi\,\mathbf{i} + \hat{\gamma}\sin\phi\,\mathbf{j} - \left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{k}.\end{aligned}$$

The cosine of the angle of the quaternion is the scalar term divided by the norm of the quaternion.

$$\vartheta_{\mathbf{yz}} = \cos^{-1}\left[\frac{\hat{\beta}\cos\phi - \hat{\alpha}\sin\phi}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\right].$$

The projection of **z** into the plane of the transformed **y** is expressed by the $\boldsymbol{Q}_{\mathbf{yz}}$ with an angle of $\pi/2$ multiplied times $\bar{\mathbf{y}}$. The **y** transform is $-\mathbf{i}\sin\phi + \mathbf{j}\cos\phi$.

$$\bar{\mathbf{z}} = \left[\frac{\hat{\gamma}\,cos\,\phi\,\mathbf{i} + \hat{\gamma}\,sin\,\phi\,\mathbf{j} - \left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{k}}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\right] * \left(-\mathbf{i}\,sin\,\phi + \mathbf{j}\,cos\,\phi\right)$$

$$= \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[cos\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{i} + sin\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{j} + \hat{\gamma}\,\mathbf{k}\right]$$

The three basis vectors for the $\bar{\mathbf{y}}$ orientation are :

$$\tilde{\mathbf{x}} = \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[\hat{\gamma}\,cos\,\phi\,\mathbf{i} + \hat{\gamma}\,sin\,\phi\,\mathbf{j} - \left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{k}\right],$$

$$\bar{\mathbf{y}} = -\mathbf{i}\,sin\,\phi + \mathbf{j}\,cos\,\phi\,,$$

$$\tilde{\mathbf{z}} = \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[cos\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{i} + sin\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{j} + \hat{\gamma}\,\mathbf{k}\right].$$

Clearly, the swing quaternion is the $\boldsymbol{T}$ that we started out with, because it swings the **y** axis to its transformed direction. Rotation about the **k** does not change the **z** axis, therefore the intermediate value of the z axis is k. We can proceed to the calculation of the spin quaternion.

$$\boldsymbol{Q}_{\text{spin}} = \frac{\tilde{\mathbf{z}}}{\mathbf{z_i}} = \frac{\frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[cos\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{i} + sin\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{j} + \hat{\gamma}\,\mathbf{k}\right]}{\mathbf{k}}$$

$$= \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[cos\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{i} + sin\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{j} + \hat{\gamma}\,\mathbf{k}\right] * -\boldsymbol{k}$$

$$= \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[\hat{\gamma} - sin\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{i} + cos\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{j}\right].$$

The rotation quaternion that transforms the original basis vectors into the transformed basis vectors is the product of the swing and spin quaternions.

$$\boldsymbol{Q}_{\mathbf{y}\to\bar{\mathbf{y}}} = \boldsymbol{Q}_{\text{spin}} * \boldsymbol{Q}_{\text{swing}} = \boldsymbol{Q}_{\text{spin}} * \boldsymbol{T}$$

$$= \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[\hat{\gamma} - sin\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{i} + cos\,\phi\left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{j}\right] * \left[cos\,\phi + sin\,\phi * \mathbf{k}\right]$$

$$= \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[\hat{\gamma}\,cos\,\phi + \left(\hat{\alpha}\,cos\,\phi + \hat{\beta}\,sin\,\phi\right)\mathbf{j} + \hat{\gamma}\,sin\,\phi\,\mathbf{k}\right].$$

The last transformation is the rotation that moves the original basis into the frame of reference for the $\bar{\mathbf{z}}$ axis. The $\bar{\mathbf{z}}$ axis was derived above and the projection of the $\bar{\mathbf{x}}$ axis is the rotation of $\bar{\mathbf{z}}$ that carries it 90° towards the $\bar{\mathbf{x}}$ axis.

$$\begin{aligned}
\bar{\mathbf{x}} &= \left(\cos\phi + \sin\phi * \mathbf{k}\right) * \mathbf{i} \\
&= \cos\phi * \mathbf{i} + \sin\phi * \mathbf{j}\,,
\end{aligned}$$

$$\bar{\mathbf{z}} = \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[\cos\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{i} + \sin\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{j} + \hat{\gamma}\mathbf{k}\right],$$

$$\begin{aligned}
\boldsymbol{Q}_{\bar{\mathbf{z}}\to\bar{\mathbf{x}}} &= \frac{\cos\phi * \mathbf{i} + \sin\phi * \mathbf{j}}{\frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[\cos\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{i} + \sin\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{j} + \hat{\gamma}\mathbf{k}\right]} \\
&= \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}} * \left(\cos\phi * \mathbf{i} + \sin\phi * \mathbf{j}\right) * -\left[\cos\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{i} + \sin\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{j} + \hat{\gamma}\mathbf{k}\right] \\
&= \frac{\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right) - \hat{\gamma}\sin\phi\,\mathbf{i} + \hat{\gamma}\cos\phi\,\mathbf{j}}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}
\end{aligned}$$

This last expression says that the $\bar{\mathbf{z}}$ axis rotated in a radial plane as it approaches $\bar{\mathbf{x}}$. That is in accord with intuition. To compute the projection of $\bar{\mathbf{x}}$ into the plane perpendicular to $\bar{\mathbf{z}}$, we apply the quaternion to rotate $\bar{\mathbf{z}}$ through 90°.

$$\begin{aligned}
\tilde{\mathbf{x}} &= \boldsymbol{Q}_{\bar{\mathbf{z}}\to\bar{\mathbf{x}}}\left(\frac{\pi}{2}\right) * \bar{\mathbf{z}} \\
&= \frac{-\hat{\gamma}\sin\phi\,\mathbf{i} + \hat{\gamma}\cos\phi\,\mathbf{j}}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}} * \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[\cos\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{i} + \sin\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{j} + \hat{\gamma}\mathbf{k}\right] \\
&= \frac{1}{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2} * \left(-\hat{\gamma}\sin\phi\,\mathbf{i} + \hat{\gamma}\cos\phi\,\mathbf{j}\right) * \left[\cos\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{i} + \sin\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{j} + \hat{\gamma}\mathbf{k}\right] \\
&= \frac{1}{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2} * \left(\hat{\gamma}^2\cos\phi\,\mathbf{i} + \hat{\gamma}^2\sin\phi\,\mathbf{j} - \hat{\gamma}\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{k}\right).
\end{aligned}$$

The basis vectors for the $\bar{\mathbf{z}}$ frame of reference are as follows.

$$\begin{aligned}
\tilde{\mathbf{x}} &= \frac{1}{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2} * \left(\hat{\gamma}^2\cos\phi\,\mathbf{i} + \hat{\gamma}^2\sin\phi\,\mathbf{j} - \hat{\gamma}\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{k}\right), \\
\tilde{\mathbf{y}} &= \sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}\,, \\
\bar{\mathbf{z}} &= \frac{1}{\sqrt{\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\gamma}^2}}\left[\cos\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{i} + \sin\phi\left(\hat{\alpha}\cos\phi + \hat{\beta}\sin\phi\right)\mathbf{j} + \hat{\gamma}\mathbf{k}\right].
\end{aligned}$$

The swing quaternion for $\bar{\mathbf{z}}$ is the ratio of $\bar{\mathbf{z}}$ to $\mathbf{z}$.

$$\boldsymbol{Q}_{\text{swing}} = \frac{\frac{1}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2+\hat{\gamma}^2}}\left[\cos\phi\left(\hat{\alpha}\cos\phi+\hat{\beta}\sin\phi\right)\mathbf{i}+\sin\phi\left(\hat{\alpha}\cos\phi+\hat{\beta}\sin\phi\right)\mathbf{j}+\hat{\gamma}\mathbf{k}\right]}{\mathbf{k}}$$
$$= \frac{1}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2+\hat{\gamma}^2}}\left[\cos\phi\left(\hat{\alpha}\cos\phi+\hat{\beta}\sin\phi\right)\mathbf{i}+\sin\phi\left(\hat{\alpha}\cos\phi+\hat{\beta}\sin\phi\right)\mathbf{j}+\hat{\gamma}\mathbf{k}\right]*-\mathbf{k}$$
$$= \frac{1}{\sqrt{\hat{\alpha}^2+\hat{\beta}^2+\hat{\gamma}^2}}\left[\hat{\gamma}-\sin\phi\left(\hat{\alpha}\cos\phi+\hat{\beta}\sin\phi\right)\mathbf{i}+\cos\phi\left(\hat{\alpha}\cos\phi+\hat{\beta}\sin\phi\right)\mathbf{j}\right].$$

to compute the intermediate value of the **x** axis, we rotate the original $\mathbf{x}=\mathbf{i}$ axis according to the rotation quaternion $\boldsymbol{Q}_{\text{swing}}$.