Modeling Rotations of Orientable Objects in Three Dimensions:
An Introduction to Quaternions and Framed Vectors

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Introduction

The Structure of Three Dimensional Space and The Attributes of Articulated Movements

We move almost effortlessly in three-dimensional space, giving minimal attention to the complex character of such movements. Indeed, we generally do not notice the intricacies of our movements, unless they are brought to our attention. However, it is easy to demonstrate that movements about a joint have some rather unexpected characteristics.

Almost all anatomical movements are rotations in joints. By combining movements in different joints, we achieve translational movements that allow us to reach and grasp. Because they are rotations, the order in which movements occur is critical to the final position of the moving body part. A series of rotational movements in different planes will introduce a concurrent twisting movement that was not a part of any of the component movements.

One can readily differentiate a right hand from a left hand, that is, body parts are orientable. Because body parts are orientable, we must keep track of orientation as well as position when explaining anatomical movements. For instance, in order to grasp a bar it is necessary to both bring one's hand up to the bar and to orient it so as to wrap one's fingers about the bar.

We have on the order of 17 degrees of freedom in the joints of our shoulder, elbow, wrist and hand, which is several times the number needed to bring the hand to any point that the hand can reach. Despite this abundance of degrees of freedom, it is still possible to place objects within reach that can not be effectively grasped by the hand. Everyone has experienced the frustration of trying to manipulate a tool in an enclosed space that allows one to reach in only from a specific direction. This is because it is critical to control the orientation of the hand as well as its position. Orientation is critical to the specification of movement in three-dimensional space.

Argument for Quaternion Analysis of Anatomical Movements

There has been interest in the anatomical community in a systematic and precise nomenclature for describing movements at joints. The system presented by MacConaill and Basmajian and most elegantly illustrated in the British Gray's Anatomy (Williams *et al.* `95, pp.

502-510) has been particularly popular. If one goes back the original source (MacConaill and Basmajian, `77), the system is published as a series of postulates about the nature of movements of joints. Let me preface this introduction by a few brief remarks on why the system proposed by MacConaill and Basmajian has not, in the last 20 years, advanced much beyond an academic exercise in descriptive anatomy.

Probably most important problem with the MacConaill and Basmajian nomenclature is that the stated principles do not grow from a more fundamental understanding of either anatomy or space. They are completely *ad hoc*, based on a systemization of anatomical observations. This is perfectly legitimate and a good first step, because they set forth a number of points that any theoretical analysis must explain, however, they ultimately appeal to an intuitive understanding of the nature of joint surfaces and movements of rigid objects in three dimensional space. It is just these intuitions that most need a solid theoretical foundation.

Secondly, their nomenclature is completely qualitative, which limits its usefulness in the precise description of movement. One may be content to start with the observation that a given movement is an impure swing, incorporating elements of both spin and pure swing, but ultimately one would like to be able to say how much of each occurred and why those amounts occurred, rather than some other combination. Most questions about anatomical movements must be considered quantitatively if we are truly to understand them. Also, most anatomical movements involve a multi-articulated systems of joints which can be understood thoroughly only if we are able to express the relations between the component movements quantitatively.

Beyond being able to express anatomical movements in a quantitative manner we would like a nomenclature of movement to be intuitive, allowing us to visualize the relationships of the components in terms of the elements used to describe them. There are methods of describing movements of rigid objects in terms of matrices and matrix multiplication, but these systems can hardly be considered intuitive. Quaternions are fundamentally operations that correspond to natural, easily visualized, movements. One can express the essence of the appropriate quaternion with a hand movement, with the thumb pointing along the axis of the rotation and the curled fingers indicating the direction and magnitude of the rotation.

As with all quantitative descriptions of movement the exact calculations utilizing quaternions are often complex and demanding, which is why we can readily use them now, whereas they would have been difficult to use before the general availability of computers. Now it is possible to express the relations between the components of a multi-articular system in a symbolic nomenclature and let a computer interpret the expressions and carry out the computation. At the symbolic level the descriptions remain intuitive.

The Ambiguity of Standard Anatomical Movements When Not in the Cardinal Planes

An important attribute of any conceptual model used to describe and analyze anatomical movements is that it be unambiguous. There is considerable ambiguity in the definitions of movements of the body parts (Williams *et al.* `95, p. 500). Such ambiguity presents major barriers to careful analysis of movement. To illustrate this ambiguity of standard nomenclature when it is taken out of the rigid context in which it is defined consider the following analysis for the gleno-humeral shoulder joint. The cardinal anatomical osteokinematic movements for the shoulder are: abduction and adduction, about a sagittal axis, moving in the frontal plane; flexion and extension, about a transverse axis, moving in a parasagittal plane; and internal and external rotation, about a longitudinal axis, moving in the transverse plane. The definitions of all these movements assume that one is starting in the anatomical position or one of the cardinal planes.

It has been cogently argued that the more logical reference system for gleno-humeral joint movements would be the plane of the scapular blade, in which case flexion would carry the arm medial as well as anterior and cephalically and abduction would carry it anterior as well as lateral and cephalically. If we use this system, then the movements are referred to the scapula rather than the body axis and one gives up the cardinal axes and planes.

For shoulder movements of the arm starting in the anatomical position, internal and external rotation in the transverse plane are qualitatively different from the other movements in that they are rotations about the longitudinal axis of the bone, whereas the other movements are rotations about a pivot point near one end of the bone. The rotation about the axis of the humerus bone is generally taken to be more essential to the concept of internal/external rotation then the rotation about a longitudinal axis of the body, because, if the arm is moved to other positions, rotations about axis of the humerus are spoken of as internal and external rotation. Once again the movement is referenced to the humerus rather than the cardinal planes and axes.

To illustrate how ambiguity of definition may cause problems when cardinal movements are not clearly defined consider the following, where we assume the original definitions given above. Suppose that your arm is swung forward 45° from the anatomical position, where your arm hangs at your side with the palm of the hand facing anterior. Now, how is abduction defined? Presumably, since abduction is rotation about a sagittal axis, 90° of abduction would swing the arm laterally and up to extend horizontally from the shoulder, midway between straightforward and straight lateral. The palm of your hand is facing forward and medially at 45° to both directions. In another example, if your arm was initially swung to 90° flexion, then 90° of adduction would leave it in 90° flexion, but rotated about its axis 90°. However, this has commonly been considered to be internal rotation, which illustrates the ambiguity of our language for describing movement.

Swing and Spin

Rotations in which the moving segment sweeps an arc in space are called swings. Flexion, extension, abduction, and adduction at the shoulder, starting in the anatomical position, would all be swings. Rotations that occur about an axis through the moving segment are called spin. Internal and external rotations at the shoulder in the given example would be spins. Swings or spins can occur about any axis. It will be shown that which is occurring is more an attribute of the object that is moving than the movement itself.

When abduction is done from the anatomical position, then it is a pure swing. When it is done from 90° flexion it is pure spin. When it is done from any position not in the mid-coronal plane or along the transverse axis, it is a combination of swing and spin, which will be called swing, conical rotation, or conical swing, because it sweeps out a conical surface. Pure swing, that is a swing that lies entirely in a single plane, and spin are opposite extremes of a continuous spectrum of conical swings.

The Importance of Symbolic Notation

Anyone who has used roman numerals to do arithmetic knows that the symbols used make a great difference in one's analytic power. Arabic numerals are so much more useful than roman numerals because they have a great deal of the structure of our number system embedded in their form and the rules for combining them. Similarly, complex logical or mathematical problems can frequently be solved by routine manipulation of symbols, once the problem is correctly formulated in a symbolic notation. This can be true of problems in joint motion, if one uses computational methods based upon the concepts briefly developed below.

Most movements of body joints are essentially rotations of body segments about each other, therefore, to describe movements in jointed structures we must symbolically express the rotations that occur and the interactions that exist between them. To do this we need a symbolic language that is adequate to talk precisely and intuitively about rotations in three dimensional space, one that allows us to describe an articulated system accurately in a symbolic format that can be manipulated to yield quantitative results. Such a symbolic language can be constructed, building on the concepts of quaternions and framed vectors. Framed vectors are arrays of vectors that express the location, extent, direction, and orientation of an orientable object. We will briefly introduce these concepts and their use in the following discussion, then develop a more thorough understanding of them through the remainder of this essay.

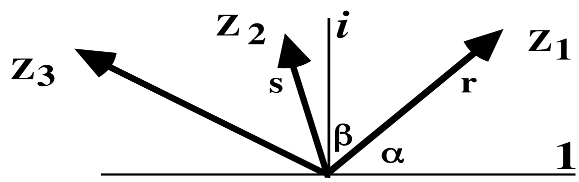
Quaternions and Framed Vectors

Quaternions : Quaternions are a generalization of complex numbers that was discovered and developed by the great Irish mathematician Sir William Rowan Hamilton a hundred and fifty years ago (Hamilton, 1899-1901; Joly, `05). As is generally known, complex numbers have a real component and an imaginary component; for example, $\boldsymbol{z} = \mathrm{a}\mathbf{1} + \mathrm{b}\mathbf{i}$; where $\mathbf{1}$ is unity for the real numbers and $\mathbf{i} = \sqrt{-1}$ is unity for the imaginary numbers and 'a' and 'b' are real numbers. Quaternions have a real component and three different imaginary components, $\boldsymbol{q} = \mathrm{a}\mathbf{1} + \mathrm{b}\mathbf{i} + \mathrm{c}\mathbf{j} + \mathrm{d}\mathbf{k}$. For both complex numbers and quaternions the real unity, $\mathbf{1}$, is generally not explicitly written, but is understood to be there.

One way of writing complex numbers allows one to view them as symbolic of rotations in two dimensions. It can be shown that

$$z = x + y\boldsymbol{i} \equiv r(\cos\alpha + \boldsymbol{i}\sin\alpha)$$

where r is a real number that equals the length of a vector and α is the angle of the complex number, that is its angle with respect to the real number axis. When we multiply the two complex numbers $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ then this may be taken to indicate that the first vector $\boldsymbol{Z}_1$ at angle α to the real axis is rotated through an angle β and made longer or shorter by a factor of s.



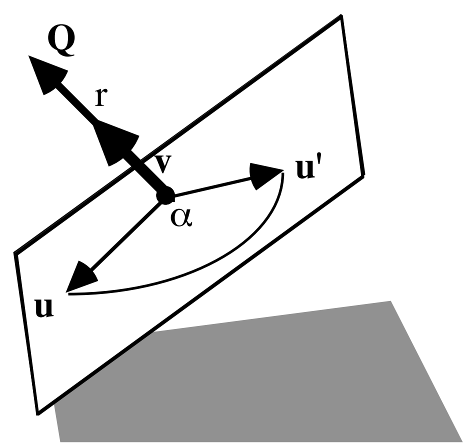
Quaternions generalize this concept by expressing the axis of rotation by a three dimensional vector, **v**, the ratio of the lengths by a multiplicative factor, r, and the angle of rotation about the axis vector by the angle of the quaternion, α, as follows:

$$\boldsymbol{Q} = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} = r(\cos\alpha + \mathbf{v}\circ\sin\alpha);$$

$$r = \sqrt{a^2 + b^2 + c^2 + d^2};$$

$$\mathbf{v} = \frac{b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}}{r};$$

$$\alpha = \cos^{-1}\left(\frac{a}{r}\right).$$



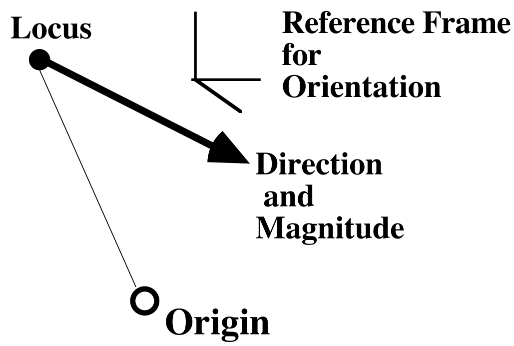
$$\mathbf{u}' = \boldsymbol{Q}\circ\mathbf{u}$$

Quaternions were used for a considerable time in theoretical physics because they are very useful for modeling rotations in three-dimensional space. Because a quaternion may be interpreted as the ratio of two arbitrary three dimensional vectors it may express the relationship between a vector before it is rotated and after it is rotated. They were also used in the theoretical description of electromagnetism, by Maxwell, and special relativity, by Einstein. However, because the symbolism and conceptual structure of quaternions was much richer than is necessary for most problems in physics and the nature of physical analysis changed, their use was gradually dropped in the early part of this century, in favor of vector analysis. Currently, quaternions are often used in orbital mechanics and three-dimensional animation, because both disciplines require the full power of quaternions to describe movements in space. We need the full conceptual richness of quaternions to adequately handle the rotations of body segments about joints, therefore they are one of the principal concepts that will used in the analysis of jointed anatomical movement.

Framed Vectors : As will be seen below, one of the characteristics of body segments that we will have to incorporate into our symbolism is orientability, the attribute of an object to be oriented in space. For instance, your hand is orientable, because it has definite dorsal and ventral surfaces, distal and proximal parts, and medial and lateral borders. Expressing orientation is done by creating vector arrays, called framed vectors, that incorporate the location, direction, magnitude, and orientation of a structure in three dimensional space,

$$\boldsymbol{f} = [\boldsymbol{\rho}, \boldsymbol{\varphi}, \boldsymbol{\lambda}, \mathbf{o}] \text{ where -}$$
$$\boldsymbol{\rho} \text{ is the location ,}$$
$$\boldsymbol{\varphi} \text{ is the direction ,}$$
$$\boldsymbol{\lambda} \text{ is the magnitude , and}$$
$$\mathbf{o} \text{ is the orientation .}$$

These arrays, which usually contain at least five vectors, are subjected to quaternion operations to model the transformations that are produced by movements in articulated systems. These components may be visualized as a vector extending from a particular point, or locus, in space, in a particular direction, for a particular distance, and associated with a specific local coordinate system, or reference frame. It will be convenient to have a term for the vector that expresses the direction and magnitude, therefore it will be called the extension vector or the extension of the framed vector.



The framed vector is the combination of a locus, an extension vector, and a reference frame. There is no necessary relationship between the direction of the extension vector and any of the vectors that compose the frame, but it is often convenient to make the frame an orthogonal set of unit vectors with one of the frame vectors aligned with the extension vector component of the framed vector.

Collections of such framed vectors may be used to describe a bone or an array of bones, a joint or a system of jointed body segments, or complex systems of bones, muscles, ligaments, and tendons. Their power lies in being able to specify the geometry of a three-dimensional object in such a way that its transformations with movement can be computed once the movements are specified as quaternions.

In this essay I will try to indicate how quaternions and framed vectors allow one to fully describe body movements and something of how they might be used to address some fairly basic questions. Most of the mechanics of manipulating quaternions will be suppressed since one must develop a substantial theoretical foundation before becoming proficient in such calculations, but the concepts can be fairly readily illustrated without detailed calculations by choosing examples that are easy to compute. We will examine very simple orientable jointed systems, so that the readers can visualize and check the results for themselves, but the real power of this approach is in the analysis of multi-joint systems with complex dynamics.

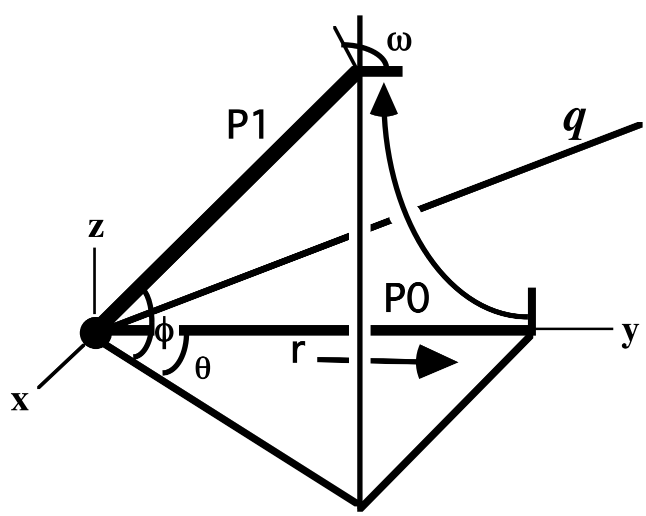
Theoretical Foundations

A Simple Anatomical Example of Rotations in Three Dimensional Space

Prior to introducing quaternions it will be helpful to examine a simple example of a rotation to illustrate where quaternions fit into the analysis of movement, particularly movements at joints. We will start with a fairly simple problem, describing the movements in a joint similar to the shoulder joint. It is commonly said that there are three degrees of freedom for movement in the gleno-humeral joint, which is in general true, but we will show that one requires at least four numbers or degrees of freedom to describe the movements of the moving bony element. Precisely what these numbers are will depend upon the coordinate system that one chooses to use.

A Simple Abstract Example of Rotations in Three Dimensional Space

Consider the following abstract situation in which an element, consisting of a straight line segment with a short perpendicular line segment at its end, is swung in a circular arc a quarter of the way around an axis, ***q***.



If the moving element is straight and pivots about one end, from an initial position, **P0**, to a new position, **P1**, a movement through a quarter of a circular arc about the axis ***q***, then the displacement of the new position relative to the initial position may be characterized by two angular displacements, symbolized here by θ and ϕ, for rotations in horizontal and the vertical planes, respectively. The distance from the pivot point to the distal end of the element (**r**) is constant, but it need not be in many practical examples. If the length of the moving element does not change in the course of the movement, then the two angular excursions move the distal end of the element over the surface of an imaginary sphere of radius **r**. Since the trajectory followed by the distal end of the moving element is not a great circle of that sphere, a third rotation, $\boldsymbol{\omega}$, occurs which rotates the bone about its longitudinal axis. This is indicated by the fact that the perpendicular segment at the distal end of the element goes from being vertical to being horizontal. It is central to all that follows that if the element had been swung through an angular excursion of θ in the xy-plane and then an excursion of ϕ in the vertical plane the perpendicular at the end of the element would be directed vertically and towards the z-axis. The path followed in moving from **P0** to **P1** determines the orientation of the perpendicular at the end of the element.

The Components of a Rotation

An element has four attributes, its location, its magnitude, the direction in which it extends, and its orientation. In this example the location, or origin of the element, is at the origin of the system, its magnitude is the magnitude of **r**, written **|r|,** and its direction is the direction of **r, r/|r|**. Its orientation is determined by the relationship between the element's axis, the vertical segment at its end, and the space that it lies within. To describe the movement of the element we must indicate how each of these four entities is changed by the movement.

The example has been chosen so that the location of the origin of the element is the pivot point, therefore the location of the element is not changed by the rotation. If the axis of the rotation had not passed through the origin of the element, then the origin of the element would have swung about the axis and thus the location of the vector would have been changed by the rotation.

The second attribute of the moving element, its magnitude, $|\mathbf{r}|$, is not changed by rotation about an axis of rotation. We can easily create examples in which the length of the element changes during a movement, but we would have to introduce other operations or compound rotations in which internal joints in the element are changing.

The third attribute, direction is clearly changed by the rotation. Precisely how will be considered later. The change in direction does not depend on any of the other attributes of the element.

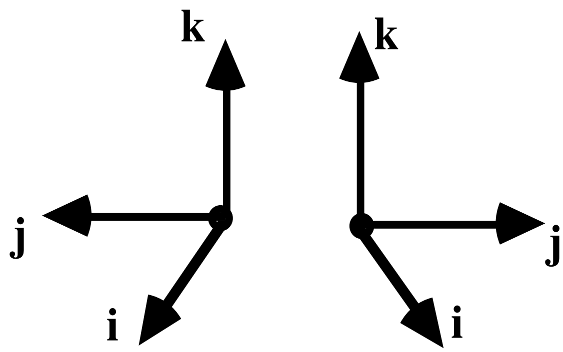
Finally, the fourth attribute of the element, it's orientation, is also changed by the rotation. In fact, the nature of the transformation is very similar to the manner in which the element's direction changes. In this example the change in orientation is indicated by ω. It also does not depend on any other attribute of the element, but it does depend on the path taken to move from the initial position to the final position. The spin ω is a rotation about the long axis of the vector. Rotations about the axis of a vector are not a property of vectors in vector analysis. It is essential that we retain it in our analysis if we are to understand movements at a joint. This last attribute is the crux of the following analysis, because it is the essence of orientability.

Orientability and Frames of Reference

In order for ω to have relevance, the object that is moving must be orientable. That is, it must have a definite front and back, top and bottom, right and left side, like your hand. One can tell a right hand from a left hand because there is a different spatial relationship between the parts of each hand. There is a clear dorsal surface and ventral surface, radial and ulnar surface. If a hand is rotated in space, then one can easily see that it has been rotated as long as the movement is not a complete circle about a single axis. If an object is orientable, then one should be able to quantitatively specify its orientation.

To completely specify orientation one needs three non-coplanar vectors, which will be called a frame of reference, reference frame, or simply a frame. The vectors that form the frame are usually chosen to be mutually orthogonal unit vectors, but neither orthogonality nor unit length is necessary attributes of a reference frame. One of the reference frame vectors might be aligned with the axis of the moving element. The second orientation vector might be aligned with a particular perpendicular to the axis of the moving element. The third vector can then be orthogonal to the other two. However, there are two possibilities for the third frame vector that point in opposite directions. One can specify one or the other based on a rule, such as, choose the vector obtained by a right-hand rule, that is, choose the direction that the right thumb points when the curled fingers move from the axial vector to the perpendicular vector. If the order of the coordinate axes is **i**, **j, k**, then the coordinate systems are as follows -

In three-dimensional space there is no way that a right-handed coordinate system can be converted into a left-handed coordinate system by any combination of translations or rotations. Notice however that one is the mirror image of the other. Because of this relationship it may be convenient to use right-handed coordinate systems to model body segments on the right side of the body and left-handed coordinate system to model the same body part on the left side of the body.

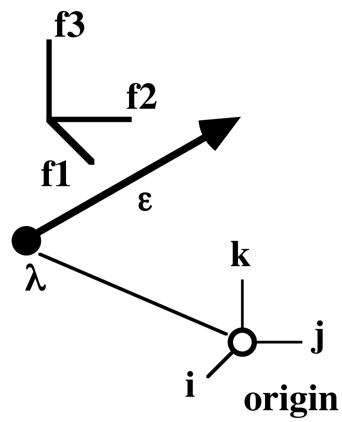


left handed **right handed**

Frames may be attached to a three dimensional structure to indicate a local orientation. Such a frame can stand in for the full detail of the structure. For instance, consider the arm hanging from the shoulder with the hand being used as an indicator of the orientation of the arm. Suppose the arm is in the anatomical position. One axis of the frame of reference might lie along the axis of the third digit, the second frame axis might be extending perpendicular to the palm in an anterior direction, and the third axis be perpendicular to the first two and extending laterally, towards the thumb. Having established this frame, it is possible to forget about the details of the hand, forearm, and arm and examine the consequences of shoulder movements by computing the manner in which the frame of reference is transformed by the specified rotations. If we wished to look at the consequences of concurrent rotations at the elbow and wrist we could attach additional frames to the humerus, the ulna, and the radius to monitor the orientation of each element in the system.

It should be noted that though a frame of reference may be said to be 'attached' to a structure it can be demonstrated that a frame is not changed by translation in space, therefore it will often be convenient to conceptually move the frame to simplify the computation. In other words, location is not an attribute of a frame of reference and it does not enter into any computations involving the frame of reference. However, it may be relevant which structure the frame is attached to, because it may or may not be appropriate to transform the frame depending on whether the structure moves during the movement.

The Components of Framed Vectors



When computing the movements of a body segment we can use vector arrays called framed vectors, which codify four attributes of the structure. The structure has a location in space, it is generally extended in the space so that it has a magnitude and direction, and it is orientable. Location is measured relative to a universal coordinate system and it may be specified as a fixed vector extending from the origin of the coordinate system to a specific point on the structure. This vector is the first component or location of a framed vector codifying the structure. The extension of the structure is indicated by a second vector that connects one specified point within the structure to another. For instance, the humerus might be specified by a vector from the center of its head to a point on its distal condyles. This second vector codes a direction and magnitude for the structure and forms the second component of the framed vector codifying the structure. If orientation is one of the relevant attributes of the structure, then one may attach a frame of reference to it. The three vectors in the frame of reference are the third, fourth and fifth components of the framed vector codifying the structure or its frame. Consequently, a framed vector is composed of five vectors; 1). a location vector (**λ**), 2) a direction and magnitude or extension vector (**ε**), and 3) a frame of reference triad of vectors (**f1**, **f2**, **f3**).

Spatial Transformations of Framed Vectors

There are three basic transformations that may be applied to an orientable structure in three dimensional space; translation, rescaling, and conical rotation. The last of these has two special cases, pure swing and pure spin. Each of the parts of the framed vector transforms differently under the three different basic spatial transformations. Translation changes only the framed vector's location. Rescaling or changing the magnitude of a structure changes only the extension of the structure therefore acts only on the second vector component. Conical rotation, or generalized rotation about an axis acts on the location of the vector, unless the axis of rotation is through its locus; on its direction, unless the axis of rotation is parallel with its direction; and always on its orientation, no matter how it is oriented relative to the axis of rotation.

Definition of Quaternions and Their Interpretation as Rotations

All body segments are orientable extended structures located in space, therefore if we are to describe the manner in which they move about the joints, we are going to have to consider how their locations, axes, and orientations interact with each other and how their attributes depend upon the movements to which they are subjected. But, how do we model such movements in a manner that will allow us to compute the consequences of particular movements? An elegant mathematical solution, which gives new insights into the process, may be quaternions, a generalization of complex numbers, discovered by Hamilton in 1843. First, the definition and some of the properties of quaternions will be discussed, then their use to represent rotations will presented.

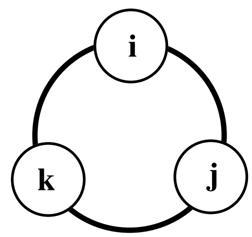
A quaternion is the sum of a real number, or scalar, and three different imaginary numbers, which together form a vector. For instance, $\boldsymbol{q} = \mathrm{a} + \mathrm{b}\mathbf{i} + \mathrm{c}\mathbf{j} + \mathrm{d}\mathbf{k}$,where ‘a’, ‘b’, ‘c’, and ‘d’ are real numbers and **i**, **j**, and **k** are different imaginary numbers, is a quaternion. Quaternions add or subtract by adding or subtracting the coefficients of like components.

$$\text{If } \boldsymbol{q}_1 = \mathrm{a}_1 + \mathrm{b}_1\boldsymbol{i} + \mathrm{c}_1\boldsymbol{j} + \mathrm{d}_1\boldsymbol{k} \text{ and } \boldsymbol{q}_2 = \mathrm{a}_2 + \mathrm{b}_2\boldsymbol{i} + \mathrm{c}_2\boldsymbol{j} + \mathrm{d}_2\boldsymbol{k}, \text{ then}$$
$$\boldsymbol{q}_{1+2} = \boldsymbol{q}_1 + \boldsymbol{q}_2 = (\mathrm{a}_1 + \mathrm{a}_2) + (\mathrm{b}_1 + \mathrm{b}_2)\boldsymbol{i} + (\mathrm{c}_1 + \mathrm{c}_2)\boldsymbol{j} + (\mathrm{d}_1 + \mathrm{d}_2)\boldsymbol{k}.$$

The rules of multiplication, which will be used extensively, are as follows:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}\,,$$
$$\mathbf{ij} = \mathbf{k} = -(\mathbf{ji})\,,$$
$$\mathbf{jk} = \mathbf{i} = -(\mathbf{kj})\,,$$
$$\mathbf{ki} = \mathbf{j} = -(\mathbf{ik})\,.$$

The last three sets of rules are easily remembered if one visualizes the following figure



and remembers that any two components multiplied in a clockwise order are the third element and the same two elements multiplied in the counter-clockwise order are the negative of the third element. The square of any element is -1.

Quaternions may be interpreted as a natural and significant extension of three-dimensional vectors. A quaternion with its scalar component equal to zero is a three dimensional vector. A quaternion with a null vector component is a real number. Quaternions are closed under multiplication, meaning that the product of any two quaternions is a quaternion. It follows that the product of any two vectors is a quaternion, which is of fundamental importance in this analysis. Since division can be represented as the product of the numerator quaternion and the inverse of the denominator quaternion, which is also a quaternion, quaternions are also closed under division.

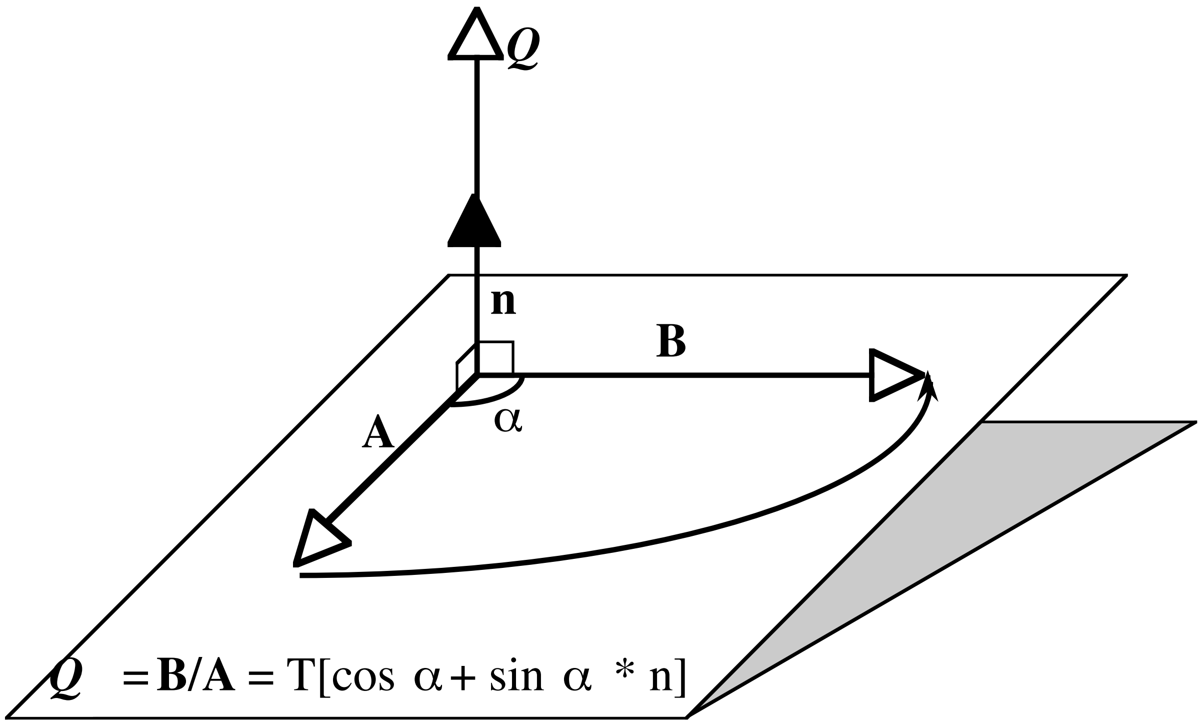
The ratio between the magnitude of the scalar component and the magnitude of the quaternion may be interpreted as the cosine of an angle, α,

$$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2 + d^2}}\,.$$

The angle a is called the angle of the quaternion and it is relevant when describing oriented elements and when computing rotations about an axis, as in the above example.

An alternative, but equivalent, way to express a quaternion is as follows -

$$\boldsymbol{Q} = \mu(\cos\alpha + \sin\alpha * \mathbf{v}), \quad \text{where}$$
μ = the magnitude of $\boldsymbol{Q}$,
α = the angle of $\boldsymbol{Q}$, and
$\mathbf{v}$ = the vector axis of $\boldsymbol{Q}$.



This is illustrated in the superjacent figure in which the vector **B** is divided by the vector **A**. **A** and **B** determine a plane to which the unit vector **n** is normal. The angle of excursion in passing from **A** to **B** is α. T is the ratio of the magnitudes of the two vectors, $T = \frac{|\mathbf{B}|}{|\mathbf{A}|}$.

It can be demonstrated that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors in three dimensional space, then their ratio, $\mathbf{v}_2/\mathbf{v}_1$, can be expressed as a quaternion $\boldsymbol{Q}$ where μ is the ratio of their lengths, α is the angle between their directions in the plane that contains both, and $\mathbf{v}$ is the unit vector normal to that plane. Therefore, a quaternion may be interpreted as the ratio of two vectors or an expression of the rotation of one vector into another, $\mathbf{v}_2 = \boldsymbol{Q} \circ \mathbf{v}_1$. It is this latter interpretation that will be particularly useful in describing rotations in joints.

For pure swings, that is swings that stay in a single plane, we can represent the rotation by a quaternion multiplication. For example - if $\mathbf{r}$ is the axial vector for the swinging armature and it rotates through an angle of α in the plane normal to $\mathbf{v}$, then the new armature vector, $\mathbf{r}'$, is given by

$$\mathbf{r}' = \boldsymbol{Q} \circ \mathbf{r}$$
$$\boldsymbol{Q} = \lambda(\cos\alpha + \sin\alpha * \mathbf{v}), \quad \text{where} \quad \lambda = 1,$$
$$\mathbf{v} = \text{the unit normal vector for the plane} \quad .$$

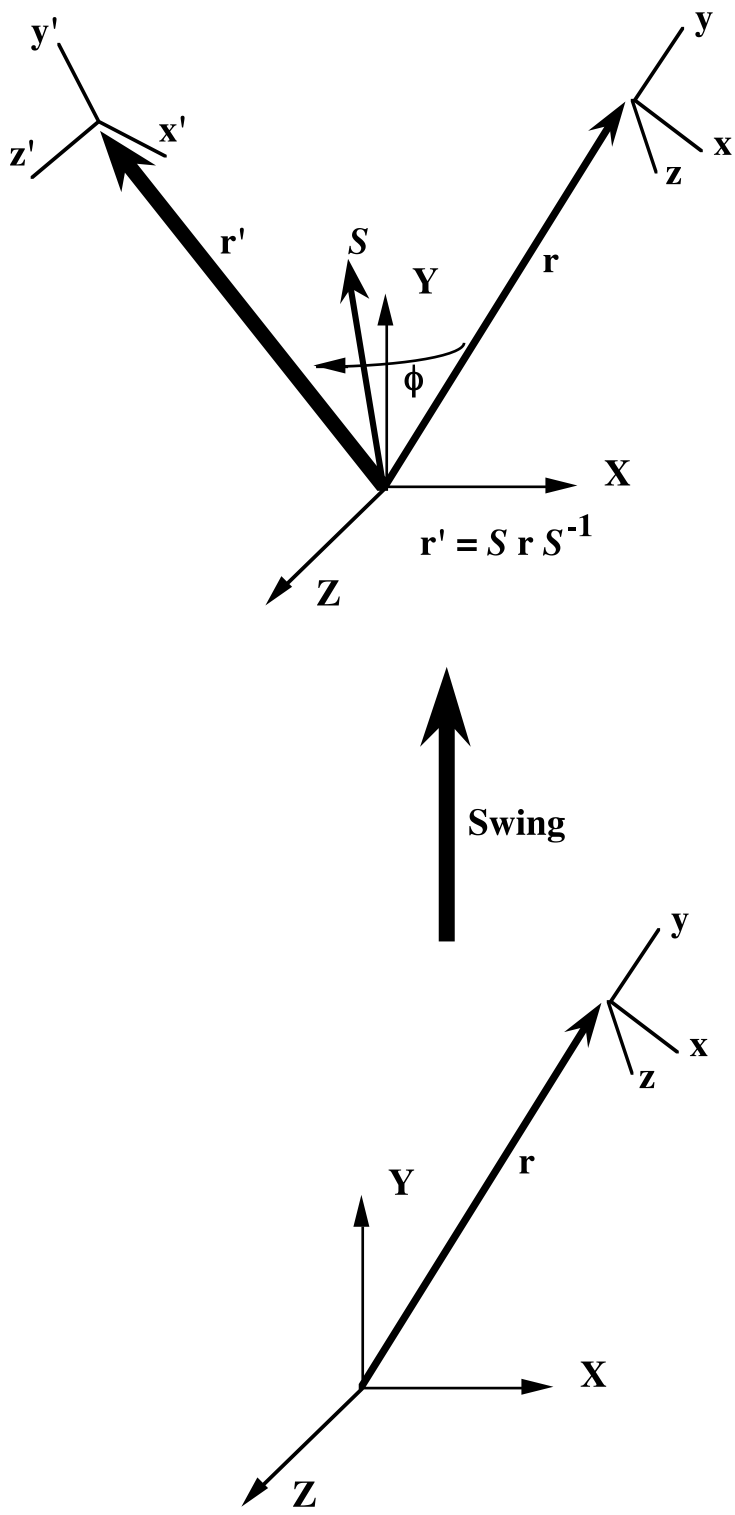
However, most swings are not pure swings, therefore this formulation is of limited usefulness for describing joint movement. In the next section we consider the more general situation when the armature rotates about an axis of rotation that is not orthogonal to its axis. This type of rotation will be called a conical rotation or swing.

Euler's Formula and General Conical Rotations

It can be demonstrated that if $\boldsymbol{Q}$ is a quaternion whose vector component, $\mathbf{r}$, represents the axis of a rotating element and $\boldsymbol{S}(\phi, \mathbf{v})$ is a quaternion with the vector component $\mathbf{v}$ and the angle ϕ, then $\boldsymbol{Q}' = \boldsymbol{SQS}^{-1}$, where $\boldsymbol{S}^{-1}$ is the inverse of $\boldsymbol{S}$, is the representation of $\boldsymbol{Q}$ if it is rotated about the axis $\mathbf{v}$ through an angle of 2ϕ. This is Euler's formula. The vector component of $\boldsymbol{Q}$ is generally not orthogonal to the vector component of $\boldsymbol{S}$, therefore this formula is used to represent the circular rotation of any vector about any axis, that is, any conical rotation. A general conical rotation is illustrated in the following figure.

Note that the object that is rotated is a vector, therefore Euler's formula is not sufficient, by itself, to calculate the transformation of an oriented object by a conical rotation. One needs a marker that is transformed by the rotation. That is why it was necessary to create frames of reference. It is the transformation of the frame of reference that allows one to follow the change in orientation. To transform a frame, $\boldsymbol{f}$, one substitutes it into Euler's formula as was done with $\mathbf{r}$.

$$\boldsymbol{f}' = \begin{Bmatrix} \mathbf{x}' \\ \mathbf{y}' \\ \mathbf{z}' \end{Bmatrix} = \boldsymbol{S} \begin{Bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{Bmatrix} \boldsymbol{S}^{-1} = \boldsymbol{S} \boldsymbol{f} \boldsymbol{S}^{-1}$$

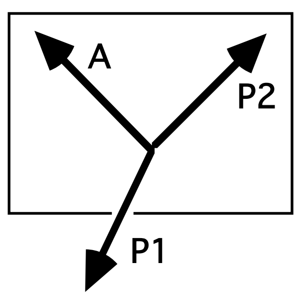


One of the fundamental properties of quaternions, one that is central to their suitability for representing rotations in three-dimensional space, is implicit in the formula $\boldsymbol{Q}' = \boldsymbol{S}\boldsymbol{Q}\boldsymbol{S}^{-1}$.

Ordinarily, one would expect that if we multiplied a quantity like $\boldsymbol{Q}$ by another quantity, $\boldsymbol{S}$, and then multiplied the result by the inverse of $\boldsymbol{S}$ we would be back to $\boldsymbol{Q}$. This is generally not true of quaternions, because they are not commutative under multiplication, that is $\boldsymbol{SQ} \neq \boldsymbol{QS}$, except under special circumstances. This is also true of rotations of orientable objects, in exactly the same way.

The Description of Rotations of Orientable Objects

We can efficiently model the movements of body segments or, more generally, the movements of orientable objects with quaternions and framed vectors. The axis of the object is represented by a single extension vector and its orientation is represented by a frame. An orientable object can be described by two vectors and a statement about whether a right handed or a left handed space is assumed, but we will generally use a set of three vectors, frequently one that lies along the axis of the object, the axial vector (**A**), one that lies along the null direction for orientation of that axis, the first perpendicular (**P1**), and a third that lies perpendicular to the first two and completes a right or left-handed coordinate system, the second perpendicular (**P2**). If these vectors are mutually orthogonal unit vectors, then they have the nice property that each is the ratio of the other two.



$$\mathbf{P2} = \frac{\mathbf{P1}}{\mathbf{A}}, \quad \mathbf{A} = \frac{\mathbf{P2}}{\mathbf{P1}}, \quad \mathbf{P1} = \frac{\mathbf{A}}{\mathbf{P2}}, \text{ for a right handed system and}$$

$$\mathbf{P2} = \frac{\mathbf{A}}{\mathbf{P1}}, \quad \mathbf{A} = \frac{\mathbf{P1}}{\mathbf{P2}}, \quad \mathbf{P1} = \frac{\mathbf{P2}}{\mathbf{A}}, \text{ for a left handed system .}$$

If the three vectors in the frame of reference are specified in their proper order, then it is not necessary to specify whether the frame is right-handed or left-handed.

Some questions can be addressed by considering only frames, but more generally one attaches a frame to a vector and both are transformed by the same operations, each coding certain attributes of the object that is being manipulated.

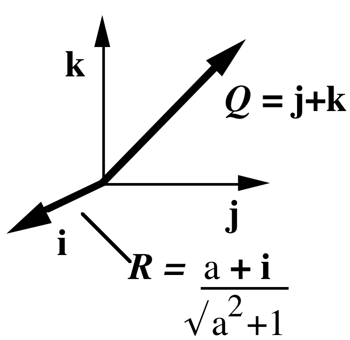
An Example of Modeling Rotations with Quaternions

Consider a very simple situation in which the rotating object is represented by a vector in the **j,k**-plane with its tail at the origin and it is rotating about an axis aligned with the **i** coordinate axis. This situation is illustrated in the following figure, where the moving element is indicated by $\boldsymbol{Q} = \mathbf{j} + \mathbf{k}$ and the unitary rotation quaternion is indicated by

$$\boldsymbol{R} = \frac{\mathrm{a} + \mathbf{i}}{\sqrt{\mathrm{a}^2 + 1}}$$

The scalar component of **R** is a variable, therefore we can look at the general example of positive rotations of any magnitude. A positive rotation is one that occurs in the direction from **j** to **k** about the **i** axis. This is according to the right hand rule, which states that the direction of positive rotations about an axis is the direction that the fingers of your right hand curl when your thumb is aligned with the axis of rotation. If we wished to consider rotations in the opposite direction, then we would use **- i** as our axis and the rotation quaternion would be

$$\boldsymbol{R} = \frac{\mathrm{a} - \mathbf{i}}{\sqrt{\mathrm{a}^2 + 1}}$$



As stated above the rotated position of the object, which remains in the **j,k**-plane is given by -

$$\boldsymbol{Q}' = \boldsymbol{R}\boldsymbol{Q}\boldsymbol{R}^{-1} = \left(\frac{1}{\sqrt{\mathrm{a}^2 + 1}}\right)^2 (\mathrm{a} + \mathbf{i})(\mathbf{j} + \mathbf{k})(\mathrm{a} - \mathbf{i})$$

$$= \left(\frac{1}{\mathrm{a}^2 + 1}\right)\left\{\left[\mathrm{a}^2 - 2\mathrm{a} - 1\right]\mathbf{j} + \left[\mathrm{a}^2 + 2\mathrm{a} - 1\right]\mathbf{k}\right\}.$$

Calculation of the new position for a wide range of rotations indicates that the new vector is the old vector rotated through two times the angle of the rotation quaternion.

If we carry out the calculation by substituting a series of values for the angle of the rotation quaternion, θ, it is easily shown by calculation that the effect of the rotation quaternion in the Euler's Formula is to rotate the vector component of the vector $\boldsymbol{Q}$ through twice the angle of the quaternion.

Calculation of Effect of Euler's Formula on $Q's$ Vector

θ	a	$\frac{a}{\sqrt{a^2+1}}$	j coefficient	k coefficient	Rotation of Q's vector
10	5.67	0.98	0.60	1.28	20.00
20	2.75	0.94	0.12	1.41	40.00
30	1.73	0.87	-0.37	1.37	60.00
40	1.19	0.77	-0.81	1.16	80.00
50	0.84	0.64	-1.16	0.81	100.00
60	0.58	0.50	-1.37	0.37	120.00
70	0.36	0.34	-1.41	-0.12	140.00
80	0.18	0.17	-1.28	-0.60	160.00
90	0.00	0.00	-1.00	-1.00	180.00
100	-0.18	-0.17	-0.60	-1.28	200.00
110	-0.36	-0.34	-0.12	-1.41	220.00
120	-0.58	-0.50	0.37	-1.37	240.00
130	-0.84	-0.64	0.81	-1.16	260.00
140	-1.19	-0.77	1.16	-0.81	280.00
150	-1.73	-0.87	1.37	-0.37	300.00
160	-2.75	-0.94	1.41	0.12	320.00
170	-5.67	-0.98	1.28	0.60	340.00

Application of the Approach in Kinesiological Analysis

Using Quaternions to Model a Classic Problem

Consider a classic observation in kinesiology. One starts with one's arm hanging straight down with the palm of the hand facing medially. One raises the arm through 90° of flexion, followed by 90° of horizontal abduction or extension, and then allows it to fall back to the original starting position through 90° of adduction. Even though at no point during the excursion was the arm rotated, the hand is now facing anterior, therefore the arm has rotated 90° during the trajectory. There is nothing magical about this set of maneuvers; one is simply demonstrating the properties of orientable objects moving through space.

One can model the series of movements as three orthogonal rotations. Let us represent the arm by three orthogonal vectors in a right-handed coordinate system. The **i** axis is taken to point laterally along the medial/lateral axis through the right shoulder, the **j** axis points anteriorly along the anterior/posterior axis, and **k** points superiorly along the rostral/caudal axis. The right arm starts pointing straight down or aligned along the **-k** axis

$$\mathbf{A} = 0 + 0\mathbf{i} + 0\mathbf{j} - \mathbf{k} = -\mathbf{k}\,.$$

The first and second perpendiculars will be taken to be pointing straight anterior from the shoulder joint ($\mathbf{P_1}$) and pointing straight lateral ($\mathbf{P_2}$), therefore

$$\mathbf{P_1} = 0 + 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k} = \mathbf{j} \text{ and } \mathbf{P_2} = 0 + 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{i}\,.$$

These three vector quaternions will be the frame for the moving element. This format, an axis and two perpendiculars that form a right-handed coordinate system will be a commonly used format, but there might be more or fewer quaternion vectors depending upon the needs of the problem. These three vectors are first rotated 90° about the **i** axis in a positive direction. Each frame vector is multiplied by the unit quaternion with an angle of 45° -

$$\boldsymbol{R}_1 = \frac{\mathbf{1} + 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}}{\sqrt{2}} = \frac{\mathbf{1} + \mathbf{i}}{\sqrt{2}}$$

and then by its inverse.

$$\begin{Bmatrix}\mathbf{A}\\ \mathbf{P_1}\\ \mathbf{P_2}\end{Bmatrix}_1 = \boldsymbol{R}_1 \circ \begin{Bmatrix}\mathbf{A}\\ \mathbf{P_1}\\ \mathbf{P_2}\end{Bmatrix}_0 \circ \boldsymbol{R}_1^{-1} = \boldsymbol{R}_1 \circ \begin{Bmatrix}-\mathbf{k}\\ \mathbf{j}\\ \mathbf{i}\end{Bmatrix} \circ \boldsymbol{R}_1^{-1} = \begin{Bmatrix}\mathbf{j}\\ \mathbf{k}\\ \mathbf{i}\end{Bmatrix}$$

Similarly, we then rotate the three frame vectors 90° about the **k** axis in a negative direction, $\boldsymbol{R}_2$ and then 90° about the **j** axis in a positive direction, $\boldsymbol{R}_3$

$$\begin{Bmatrix}\mathbf{A}\\ \mathbf{P_1}\\ \mathbf{P_2}\end{Bmatrix}_2 = \boldsymbol{R}_2 \circ \begin{Bmatrix}\mathbf{A}\\ \mathbf{P_1}\\ \mathbf{P_2}\end{Bmatrix}_1 \circ \boldsymbol{R}_2^{-1} = \boldsymbol{R}_2 \circ \begin{Bmatrix}\mathbf{j}\\ \mathbf{k}\\ \mathbf{i}\end{Bmatrix} \circ \boldsymbol{R}_2^{-1} = \begin{Bmatrix}\mathbf{i}\\ \mathbf{k}\\ -\mathbf{j}\end{Bmatrix}, \quad \boldsymbol{R}_2 = \frac{\mathbf{1}-\mathbf{k}}{\sqrt{2}};$$

$$\begin{Bmatrix}\mathbf{A}\\ \mathbf{P_1}\\ \mathbf{P_2}\end{Bmatrix}_3 = \boldsymbol{R}_3 \circ \begin{Bmatrix}\mathbf{A}\\ \mathbf{P_1}\\ \mathbf{P_2}\end{Bmatrix}_2 \circ \boldsymbol{R}_3^{-1} = \boldsymbol{R}_3 \circ \begin{Bmatrix}\mathbf{j}\\ \mathbf{k}\\ \mathbf{i}\end{Bmatrix} \circ \boldsymbol{R}_3^{-1} = \begin{Bmatrix}-\mathbf{k}\\ \mathbf{i}\\ -\mathbf{j}\end{Bmatrix}, \quad \boldsymbol{R}_3 = \frac{\mathbf{1}+\mathbf{j}}{\sqrt{2}}.$$

If we compare the last set of frame vectors to the first set, then it is apparent that the arm frame of reference has returned to its original position, pointing straight down, but the two perpendiculars have rotated 90° laterally about the axis of the original arm frame. This is precisely what happens if we do this set of maneuvers with a real arm. We can represent the entire set of rotations by the following expression.

$$\begin{Bmatrix}\mathbf{A}\\ \mathbf{P_1}\\ \mathbf{P_2}\end{Bmatrix}_F = \boldsymbol{R}_3 \circ \boldsymbol{R}_2 \circ \boldsymbol{R}_1 \circ \begin{Bmatrix}\mathbf{A}\\ \mathbf{P_1}\\ \mathbf{P_2}\end{Bmatrix}_0 \circ \boldsymbol{R}_1^{-1} \circ \boldsymbol{R}_2^{-1} \circ \boldsymbol{R}_3^{-1} = \boldsymbol{R}_3 \circ \boldsymbol{R}_2 \circ \boldsymbol{R}_1 \circ \begin{Bmatrix}-\mathbf{k}\\ \mathbf{j}\\ \mathbf{i}\end{Bmatrix} \circ \boldsymbol{R}_1^{-1} \circ \boldsymbol{R}_2^{-1} \circ \boldsymbol{R}_3^{-1} = \begin{Bmatrix}-\mathbf{k}\\ \mathbf{i}\\ -\mathbf{j}\end{Bmatrix}.$$

This is a fairly simple situation, where we can actually do the calculations in our head as we go through the maneuvers, but this same methodology can be applied for any set of rotation axes through any angular excursions. Once we start using non-orthogonal axes and angular excursions other than right angles, the computations without quaternions become extremely complex and it becomes increasingly difficult to reason geometrically.

Conjoint Movements of Tilted Orientable Objects: Another familiar problem, from the biomechanics of the spine, involves the conjoint rotation that occurs when a spinal vertebra is side-flexed. It is observed that if the vertebra is aligned so that an axis perpendicular to the superior and inferior faces is vertical and then it is rotated about an horizontal axis in its mid-sagittal plane, that is side-flexed, then there is no conjoint rotation. However, if the vertebra is tipped forward, so that the top surface faces superior and anterior and then rotated about the same horizontal axis, then the side-flexion is accompanied by a conjoint rotation of the vertebra about a vertical axis so that the anterior face of the vertebra faces in the direction opposite to the direction of the side-flexion. The spine of the vertebra is pointing in the direction of the side-flexion.

To model the vertebra let us make the axis vector the perpendicular to the dorsal surface of the vertebral body. The first perpendicular to the axis is in the mid-sagittal plane, that is, perpendicular to the anterior surface, and the second perpendicular points to the left. These three vectors define an orthogonal right-handed coordinate system attached to the vertebra. If the vertebra is sitting on the inferior surface of its body, then the frame will be -

$$\mathbf{D} = \begin{Bmatrix} \mathbf{k} \\ \mathbf{i} \\ \mathbf{j} \end{Bmatrix}$$

If we rotate the vertebra about a horizontal midsagittal axis then the rotation quaternion will be $\boldsymbol{R} = \frac{\mathbf{1} + \mathbf{i}}{\sqrt{2}}$. Carrying out the calculation we obtain

$$\mathbf{D}' = \boldsymbol{R} \circ \mathbf{D} \circ \boldsymbol{R}^{-1} = \begin{Bmatrix} -\mathbf{j} \\ \mathbf{i} \\ \mathbf{k} \end{Bmatrix}.$$

Since the first perpendicular, the vector perpendicular to the anterior surface of the vertebra, continues to lie along the same axis, which is straight anterior in our example, there is not rotation of the vertebra about a vertical axis.

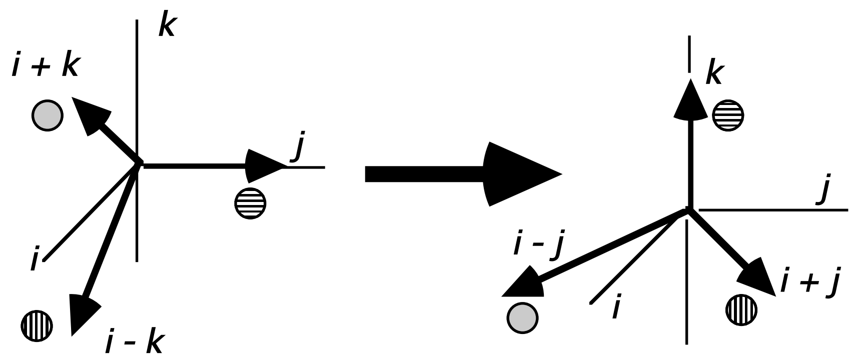
Now we turn to the anteriorly tipped vertebra. Its frame, if it is tilted 45° anterior will be

$$\mathbf{D} = \begin{Bmatrix} \lambda \circ \mathbf{i} + \lambda \circ \mathbf{k} \\ \lambda \circ \mathbf{i} - \lambda \circ \mathbf{k} \\ \mathbf{j} \end{Bmatrix}; \lambda = \frac{1}{\sqrt{2}}.$$

The λ is just a constant factor to make the length of the descriptor vectors unity. The rotation quaternion is the same as in the previous example, therefore the new descriptor for the vertebra is

$$\mathbf{D}' = \boldsymbol{R} \circ \mathbf{D} \circ \boldsymbol{R}^{-1} = \begin{Bmatrix} \lambda \circ \mathbf{i} - \lambda \circ \mathbf{j} \\ \lambda \circ \mathbf{i} + \lambda \circ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}.$$

Examination of the first perpendicular indicates that the anterior-posterior axis of the vertebra has rotated so that it faces in the direction opposite to that in which the vertebra was side-flexed.



This is illustrated in the figure in which the axis is indicated by the hatched ball, the first perpendicular by the ball with the vertical stripes, and the second perpendicular by the ball with the horizontal stripes.

If we tilt the vertebra posteriorly, then its reference frame becomes

$$\mathbf{D} = \begin{Bmatrix} -\lambda \circ \mathbf{i} + \lambda \circ \mathbf{k} \\ \lambda \circ \mathbf{i} + \lambda \circ \mathbf{k} \\ \mathbf{j} \end{Bmatrix}.$$

and if it is rotated about the same horizontal axis then the frame is

$$\mathbf{D}' = \boldsymbol{R} \circ \mathbf{D} \circ \boldsymbol{R}^{-1} = \begin{Bmatrix} -\lambda \circ \mathbf{i} - \lambda \circ \mathbf{j} \\ \lambda \circ \mathbf{i} - \lambda \circ \mathbf{j} \\ \mathbf{k} \end{Bmatrix}$$

Notice that the $\mathbf{j}$ terms have the same sign, therefore the vertebra is rotated in the same direction as the direction of the side-flexion.

Supposing that we rotate the vertebra about a vertical axis, rather than a horizontal axis. We know that if the vertebra is vertically aligned, then there will be no side-flexion, but if it is tilted, then there will be a conjoint side-flexion. The results of rotation about a vertical axis, $\boldsymbol{R} = \frac{1 + \boldsymbol{k}}{\sqrt{2}}$ for the three situations considered above are summarized below.

$$\text{Vertically Aligned} \quad \begin{Bmatrix} \mathbf{k} \\ \mathbf{i} \\ \mathbf{j} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \mathbf{k} \\ \mathbf{j} \\ -\mathbf{i} \end{Bmatrix}$$

$$\text{Anteriorly Tilted} \quad \begin{Bmatrix} \lambda \circ \mathbf{i} + \lambda \circ \mathbf{k} \\ \lambda \circ \mathbf{i} - \lambda \circ \mathbf{k} \\ \mathbf{j} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \lambda \circ \mathbf{j} + \lambda \circ \mathbf{k} \\ \lambda \circ \mathbf{j} - \lambda \circ \mathbf{k} \\ -\mathbf{i} \end{Bmatrix}$$

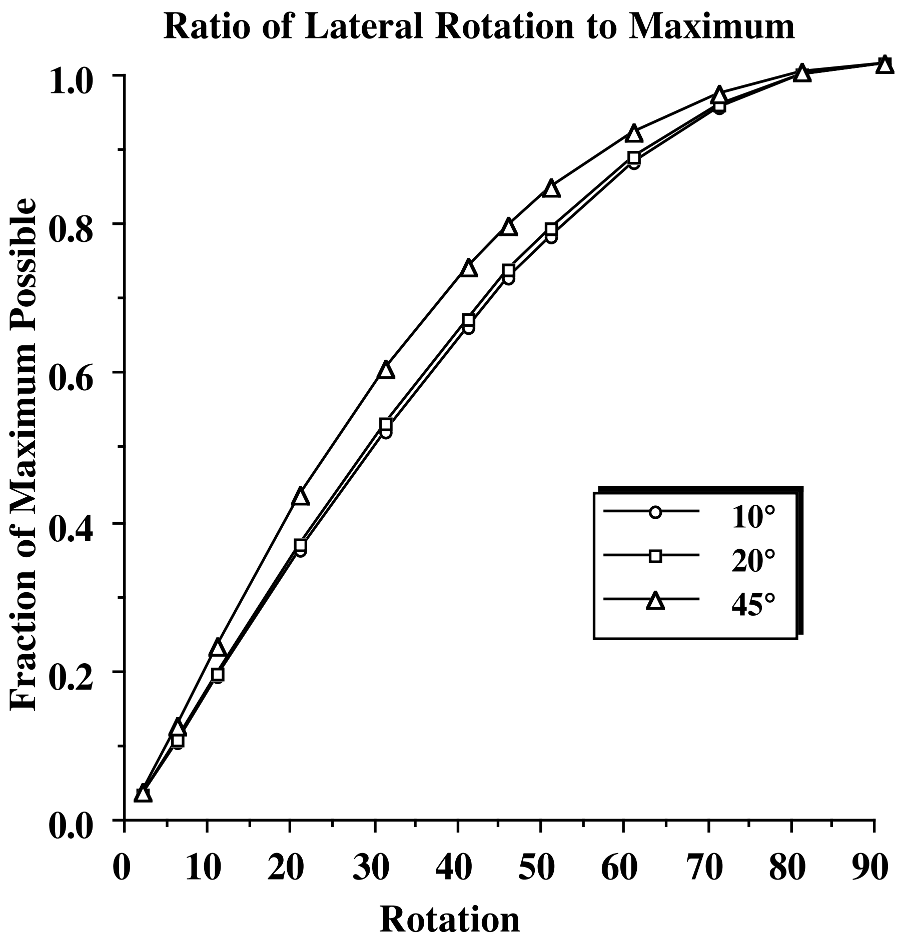
$$\text{Posteriorly Tilted} \quad \begin{Bmatrix} -\lambda \circ \mathbf{i} + \lambda \circ \mathbf{k} \\ \lambda \circ \mathbf{i} + \lambda \circ \mathbf{k} \\ \mathbf{j} \end{Bmatrix} \Rightarrow \begin{Bmatrix} -\lambda \circ \mathbf{j} + \lambda \circ \mathbf{k} \\ \lambda \circ \mathbf{j} - \lambda \circ \mathbf{k} \\ -\mathbf{i} \end{Bmatrix}$$

Note that there is a side-flexion in the instances with tilt. When the tilt is anterior, the rotation and the side-flexion are in the same direction, and when the tilt is posterior, the rotation and side-flexion are in opposite directions. Because, for anterior tilt, rotation and side-flexion are in the same direction when rotation is the primary movement and in opposite directions when side-flexion is the primary movement and *vice versa* for posterior tilt, it follows that the final configuration depends critically upon which movement is the principal movement.

For these examples, large angles for tilt and rotation were used to made it possible to check the results of the calculations by geometric reasoning and it made the effects large so that there was no doubt of the direction of the changes. Let's briefly consider a calculation in which the tilt and rotations are more typical of one might see in a real spine. Consider the following, the vertebra is tilted 10° anterior relative to the axis of the side-flexion and we side-flex 10°. We will use quaternions with unitary vector components to make the problem easier to formulate. The axis vector in the frame of reference will be $\mathbf{A} = 0.17365\ \mathbf{i} + 0.98481\ \mathbf{k}$, $\mathbf{P_1} = 0.98481\ \mathbf{i} - 0.17365\ \mathbf{k}$, and $\mathbf{P_2} = \mathbf{j}$. The rotation quaternion will be $\boldsymbol{R} = 0.99619 + 0.08716\ \mathbf{i}$. Therefore the frame for the rotated vertebra will be

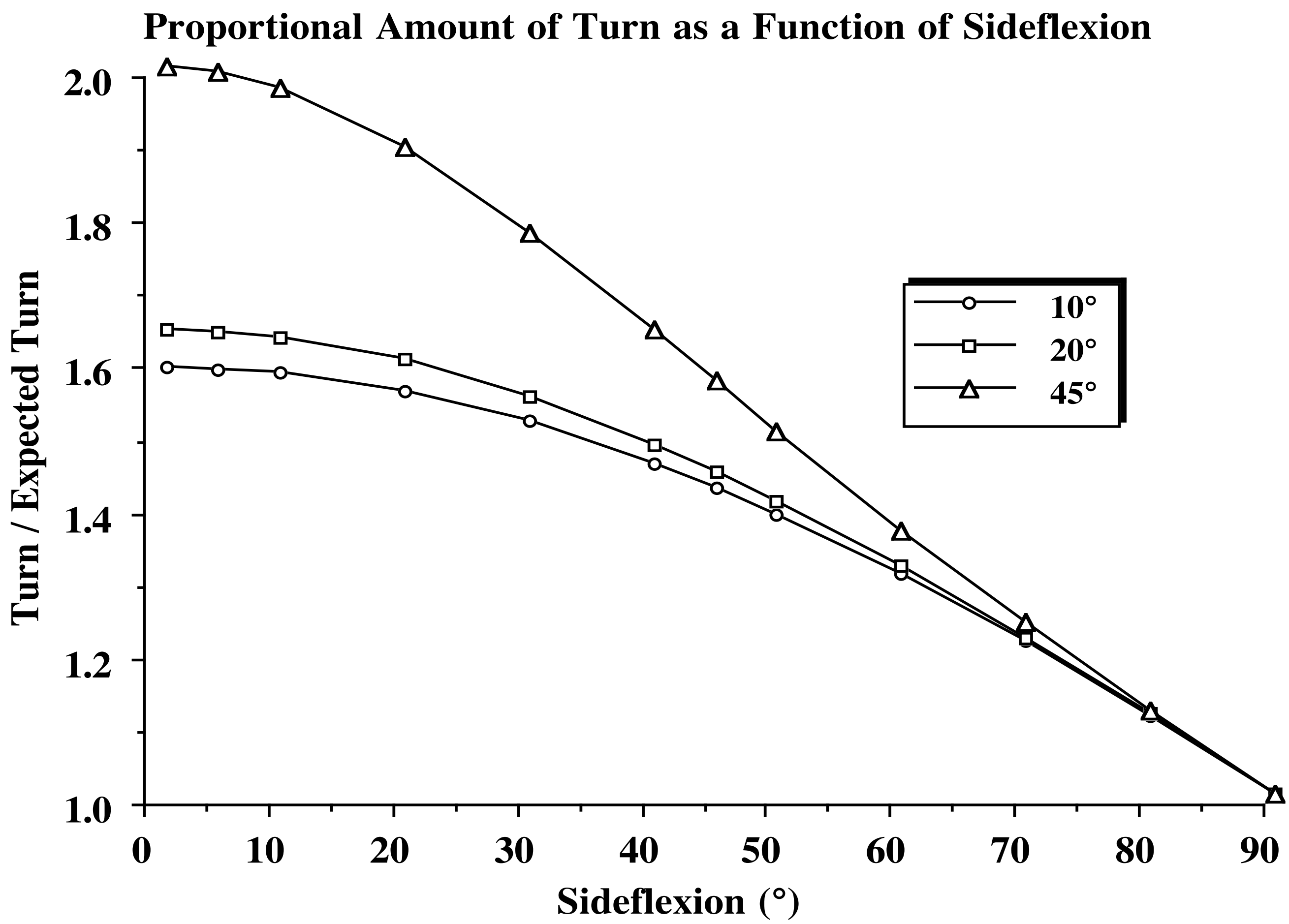
$$\begin{Bmatrix} \mathbf{A} \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{Bmatrix} = \begin{Bmatrix} 0.17365\ \mathbf{i} + 0.98481\ \mathbf{k} \\ 0.98481\ \mathbf{i} - 0.17365\ \mathbf{k} \\ \mathbf{j} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \mathbf{A}' \\ \mathbf{P}_1' \\ \mathbf{P}_2' \end{Bmatrix} = \begin{Bmatrix} 0.17365\ \mathbf{i} - 0.17102\ \mathbf{j} + 0.96985\ \mathbf{k} \\ 0.98481\ \mathbf{i} + 0.03016\ \mathbf{j} - 0.17101\ \mathbf{k} \\ 0.98481\ \mathbf{j} + 0.17366\ \mathbf{k} \end{Bmatrix}$$

This expression looks intimidating, but one can see from the $\mathbf{P_2}'$ term that the side-flexion is 10° and by the ratio of the **i** and **j** components in the $\mathbf{P_1}'$ component that the rotation in the horizontal plane is 1.73° = 1° 44′. The rotation seems small but remember that the maximum that is possible with a 10° anterior tilt is 10°, which occurs with 90° side-flexion. Therefore the rotation produced by 10° of side flexion is 11.1% of 90°, which is 17.3% of the maximum possible.



If one does the same calculation for a variety of anterior tilts of the frame of reference and for amounts of side-flexion ranging from 1° to 90°, then the relationship between the amount of turn that one observes and the maximum amount that can occur for that amount of anterior tilt looks like the above figure. If we call the rotation that occurs about a vertical axis 'turn' then turn is disproportionate to the amount of side-flexion. The relative amount of turn for a given amount of side-flexion is greater for more anteriorly tilted frames of reference.

One might expect that if the frame were side-flexed 9°, that is 10% of 90°, then the amount of concurrent turning would be 10% of the amount that occurs with 90° of side-flexion. However, the actual amount of concurrent turning is about 16% of the maximum when the tilt is 10°. If we systematically make this comparison the relationship looks like the above figure. For small amounts of side-flexion there is a disproportionate amount of concurrent turning. If the frame of reference's anterior tilt is 45°, then the amount of concurrent turn with side-flexion is twice expectation for small angles of side-flexion.



Keep in mind though that the actual amount of turning for small amounts of anterior tilt is very small. From the above calculation, a frame that is anteriorly tilted 10° and rotated 10° will turn only 1.75°, which is probably undetectable in most realistic systems of measurement. However, such small movements may become detectable given long lever arms and systems in which similar movements are occurring in several linked joints, such as a spinal column. These calculations have yet to be made.

The calculations necessary to study a multi-jointed system would much more complex than those illustrated here, because they required the use of arrays of framed vectors, keeping detailed accounting of each element in the array, and additional operational rules that have not been introduced in this fairly simplistic introduction. Still such a calculation is relatively straightforward in the analytic system described in this essay, using framed vectors and quaternion operations. The greatest challenge lies in writing the system down in a complete form so that the computer can carry out the calculations.

Fryette's Laws of Physiologic Spinal Motion

Early in the 1900's an osteopathic physician formulated three observations that have been essentially validated over the years and come to be called "laws". They are as follows.

I. If the vertebral segments are in the neutral (or easy normal) position without locking of the facets, rotation and sidebending are in opposite directions. This is called type I motion.
II. If the vertebral segments are in full flexion or extension with the facets locked or engaged, rotation and sidebending are to the same side. This is called type II motion.
III. If motion is introduced into a vertebral segment in any plane, motion in other planes is reduced.

Given the analysis of the effect of tilt relative to the axis of rotation on the relationship between side-flexion and turn there are some points that we can make on the theoretical basis of Fryette's laws. Sidebending is the same as side-flexion and the rotation referred to in the laws is what has been called turn here. Anterior or posterior tilt occurs is a plane that also contains the axis of rotation, side-flexion is rotation in an orthogonal plane, and turn is rotation in the third orthogonal plane.

Type I motion occurs in the situation where the vertebral body is anteriorly tilted relative to the axis of rotation, because there is both side-flexion and turn and they occur in opposite directions. Such motion occurs in the lumbar spine because the vertebral bodies are anteriorly tilted relative to an axis parallel with the transverse axis of the body. When the lumbar spine is side-flexed from neutral position it also rotates about a transverse axis. Consequently, the turn associated with a right side-flexion must be to the left.

That leaves the question of why the relationship is reversed when the lumbar spine is either in endrange flexion or endrange extension. The observation that the side-flexion and turn occur in the same direction tells us that the vertebral body must be posteriorly tilted relative to the axis of rotation. One can confirm that this is the case by examining the lumbar spine. Side-flexion at either endrange flexion or extension means that the axis of rotation must run through one of the facet joints, because the facet joint acts as a pivot for the rotation as the opposite facet moves towards a more centered placement. The orientation of the lumbar facets is such that the axis of rotation passes inferiorly and laterally relative to the body of the vertebra, which means that the vertebral body is effectively posteriorly tilted relative to the axis of rotation. Consequently, as the vertebras side-flexes to the right about an axis through the fixed facet joint it also turns to the right.

The third law is probably largely a consequence of the fact that there is the least stress on ligaments and discs, therefore the greatest available range of motion in any direction, when the joint is in midrange or neutral position. Bones are also more apt to collide if they are near the endrange of the joint between them.

Fryette's laws arise from the particular anatomy of the spine and the properties of rotations in three-dimensional space. If the anatomy were different, then the relationships would be different. One can easily imagine situations in which the first law would not apply. All that is necessary is to define side-flexion so that the vertebral bodies are posteriorly tilted relative to the axis of rotation. This exercise shows how certain observations about the motion of body segments are a consequence of the geometry of the situation and while dependent on certain attributes of their anatomy they do not require special properties of the involved structures. The structure imposes constraints on the motion by determining the location and direction of the axis of rotation.

Summation

It has been possible to address only a few points concerning the benefits of using quaternions to model the rotations of orientable objects in three dimensional space. To go much further would require laying a broader foundation and delving deeper into the mechanics of manipulating quaternions. However, I hope that enough has been presented here to illustrate the potential of using quaternions to describe and compute movements in space.

I visualize this approach being useful in two rather different levels of analysis. First, for the detailed analysis of problems in movement where one must create precise descriptions of the elements involved and their movements and calculate the consequences of various changes in the configuration of the elements or in the movements. Quaternions and framed vectors provide a very intuitive means of dealing with such problems. Secondly, the exercise of defining the configuration and movements of orientable objects in terms of quaternions and frames of reference should lead to a more precise language that can be applied more effectively to the description of problems and provide the tools to visualize the consequences of maneuvers. For instance, one should routinely utilize the axis of rotation to describe rotations, because doing so is much more efficient and precise than such terms as internal and external rotation. Once one has developed the habit of precise description of rotations, many problems can be solved approximately or in principle, without detailed calculation. I feel that quaternions and framed vectors provide an efficient tool for describing the situations that arise in kinesiology and biomechanics and the trouble of gaining some familiarity with them will be repaid handsomely in the analytic power that they provide.

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