

Geometrical Anatomy and Anatomical Movements:
An Introduction to the Use of Quaternions and Framed Vectors in the
Description of Anatomical Rotations

Thomas Langer

Introduction to Quaternion Analysis

We normally move seemingly effortlessly, reaching out to pick up a glass, walking down the street, or looking at a passing friend. We do these things without consciously planning the movement or exerting a conscious effort. It is only when the automatic processes that usually produce these actions fail that we notice how much effort goes into their production. For instance, when a person has a stroke and becomes paretic in an arm and/or a leg, they suddenly find their limb extremely heavy and difficult to manage. When we move a paralyzed limb for someone, it is surprising how much heavier it feels than our own arm or leg. This is largely because we have lost the automatic counter-balancing and control that the body normally provides. These controls are so automatic and unconscious that we are usually completely unaware of them.

When we study the nature of anatomical movements and their control, we also find it extremely effortful to capture the movements in a manner that we can analyze and manipulate. We discover that there is a great deal of complexity and unexpected subtlety in even the simplest natural movements in three-dimensional space. In this essay, we are going to largely leave aside the production of the movement and look at the movement itself. It may be premature to become too engrossed in the control of movements until we understand what it is that is being controlled.

Initially, it may seem that there is very little to the description of movements, but a deeper examination raises a great many interesting and challenging problems. In this essay we will just touch on a number of these problems. To explore the problems in depth would require a level of analysis that is not possible in an introductory essay, but it is possible to give something of the flavor of their solution with comparatively few concepts and a little computation.

Let us start with the observation that almost all anatomical movements of the limbs, spine, head and eyes occur in joints. Almost all movements in joints are rotations. Therefore, if we are to understand anatomical movement we must understand rotations. We will find that rotations in three-dimensional space have some unexpected attributes that have profound implications for anatomical movements.

Introduction to Quaternion Analysis

The nature of movement in a particular joint is contingent upon the geometry of the joint and the shapes of the bones that participate in the joint will have considerable influence upon the nature of the generated movement. For instance, the angle between the head of the humerus and its shaft converts a spinning movement in the shoulder joint into a swinging movement of the arm.

Movements are produced by the concerted action of muscles that attach to the bones that participate in the joint, therefore the geometry of the muscle attachments with respect to the joint will be relevant to understanding the movements that occur as a result of muscle actions.

In what follows we will consider how the geometrical anatomy of a joint and the nature of three-dimensional space determine the nature of anatomical movements. To do this, we need to develop some analytic tools that allow the precise description of anatomical movement. Two of the most fundamental tools that will be developed and used extensively are the concepts of quaternions and orientation.

Quaternions are a type of mathematical object that has been known since the mid-1800's, but little studied in recent years. They have certain attributes that make them ideal for modeling rotations.

Orientation is a concept that is generally understood in an intuitive, qualitative, way, but which will be defined in a mathematical construct that allows us to manipulate it quantitatively and obtain very precise answers to questions about movement in particular circumstances.

Quaternions

Complex numbers

Sir William Rowan Hamilton discovered quaternions while trying to generalize the concept of complex numbers to three dimensions. It may be remembered, from high school algebra, that complex numbers are a combination of a real number, $a\mathbf{1}$, and an imaginary number, $b\mathbf{i}$, such that

$$c = a\mathbf{1} + b\mathbf{i}.$$

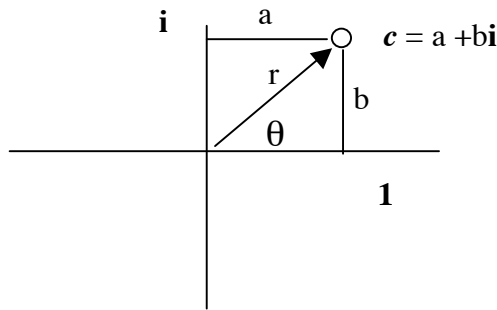
Introduction to Quaternion Analysis

The **1** in the first term is unity in the real number system and **i** is unity in the imaginary numbers. Generally, we do not write the real unity, because it is understood to be there unless otherwise indicated. It was necessary to invoke the imaginary numbers to solve equations that required the square root of a negative number, such as –

$$\begin{aligned}x^2 + 1 &= 0 \\x^2 &= -1 \\x &= \sqrt{-1} = \mathbf{i}\end{aligned}$$

One of the attributes of imaginary numbers is that their multiplication is a convenient way of modeling rotations in two dimensions. To see this one needs to represent complex numbers in a different, but equivalent, format.

First we note that complex numbers can be graphically represented as points in a two-dimensional plot in which the real number component is represented by a distance along a horizontal axis and the imaginary component is represented by a distance along a vertical axis. The complex number is the point that lies at the sum of the two distances.



In general, the sum of two complex numbers is the sum of their similar components.

$$\begin{aligned}\mathbf{c}_1 + \mathbf{c}_2 &= (\mathbf{a}_1 + \mathbf{b}_1\mathbf{i}) + (\mathbf{a}_2 + \mathbf{b}_2\mathbf{i}) \\&= (\mathbf{a}_1 + \mathbf{a}_2) + (\mathbf{b}_1 + \mathbf{b}_2)\mathbf{i}\end{aligned}$$

Note that the point may also be interpreted as the point at the end of a line that extends radially from the origin of the coordinate system. We can write down the conversion formulae by inspection of the graph.

Introduction to Quaternion Analysis

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \cos^{-1}\left(\frac{a}{r}\right) \quad ;$$

$$a = r * \cos\theta \quad \text{and} \quad b = r * \sin\theta \quad ;$$

therefore

$$c = a + b\mathbf{i} = r * (\cos\theta + \mathbf{i} * \sin\theta)$$

We call r the magnitude of the complex number and θ its angle.

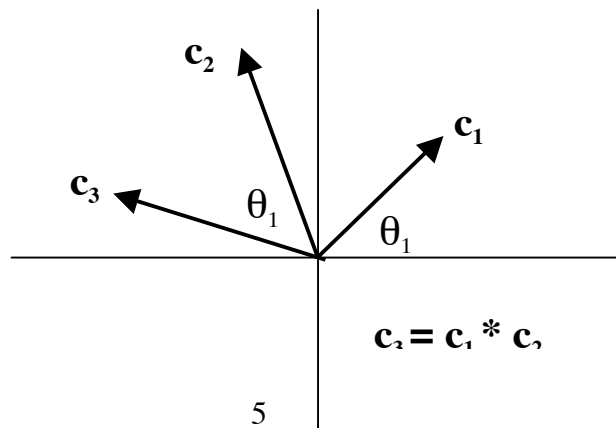
At this point we note that the product of two complex numbers is another complex number. The product is the algebraic product of the two numbers, remembering that \mathbf{i} is the square root of minus one.

$$\begin{aligned} c_1 * c_2 &= (a_1 + b_1\mathbf{i}) * (a_2 + b_2\mathbf{i}) \\ &= a_1a_2 + \mathbf{i} * (a_1b_2 + a_2b_1) + b_1b_2 * \mathbf{i}^2 \\ &= (a_1a_2 - b_1b_2) + \mathbf{i} * (a_1b_2 + a_2b_1) \end{aligned}$$

If we do the same product, but using the trigonometric format then the nature of the result is much more easily appreciated.

$$\begin{aligned} c_1 * c_2 &= r_1(\cos\theta_1 + \mathbf{i} * \sin\theta_1) * r_2(\cos\theta_2 + \mathbf{i} * \sin\theta_2) \\ &= r_1 * r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + \mathbf{i} * (\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1) \right] \\ &= r_1 * r_2 \left[(\cos(\theta_1 + \theta_2)) + \mathbf{i} * \sin(\theta_1 + \theta_2) \right]. \end{aligned}$$

We can see that the product of two complex numbers gives a complex number that has a magnitude that is the product of their magnitudes and an angle that is the sum of their angles. If one or both of the complex numbers has a magnitude of 1.0, then the multiplication may be viewed as a model of rotation in a plane. In fact, this attribute of complex numbers is used frequently in physics to model wave phenomena and oscillations.



Introduction to Quaternion Analysis

In the figure the complex number \mathbf{c}_1 has a magnitude of 1.0 and an angle of θ_1 . As θ_1 changes it causes the complex number \mathbf{c}_3 to rotate through the angle θ_1 relative to \mathbf{c}_2 . Since the magnitude of \mathbf{c}_1 is unity, the magnitude of \mathbf{c}_3 is the same as that for \mathbf{c}_2 . Consequently, \mathbf{c}_1 effectively rotates \mathbf{c}_2 to produce \mathbf{c}_3 .

There is nothing in the definition of complex numbers or of the operations that are defined on them that will cause the sum or product of two complex numbers to depend upon the order in which they appear in the expressions. This may seem like a trivial observation, but it was a major stumbling block in the discovery of quaternions, because it is not true of them.

Hamilton Extends Complex Numbers to Three Dimensions

We have taken some time to review the complex numbers, because for many people complex numbers are not fresh in their memory or they may not have encountered them before. However, some understanding of complex numbers helps in understanding quaternions. Hamilton, who was one of the greatest mathematicians of his time, or of any time, spent 30 years trying to find a logically consistent way of extending the complex numbers to three dimensions. When he did find a way, it was quaternions.

There were several factors that were unexpected and therefore made the discovery of quaternions more difficult. First, one might expect that since it takes two numbers to represent rotations in two dimensions, it would take three to do so in three dimensions. It actually takes four, for reasons that will be elaborated later. Second, since the order of multiplication is irrelevant to the result for complex numbers one might expect that the extension of complex number to three dimensions would be the same, but it is not. The order of multiplication of quaternions is highly relevant. Except in very special circumstances, the result of multiplying two quaternions is different if the order is reversed. At the time that Hamilton discovered quaternions there was not any other number system that had that property. After he established that commutativity of multiplication was not a necessity of valid number systems, there was an explosion of creation of number systems that lacked one or more of the properties of real or complex numbers. It was the beginning of what is now called modern algebra.

Introduction to Quaternion Analysis

As disconcerting as the lack of commutativity of multiplication of quaternions was, it turned out to also be a property of rotations in three dimensions. It is easily demonstrated that for sequences of two rotations that do not occur in the same plane, reversing the order of the rotations produces quite different outcomes. This may be seen by taking a book and laying it on a table front up and appropriately oriented for reading. Now rotate it through 90° so that it is lying on its spine by rotating it about its long axis and then rotate it through 90° again, but about an axis that extends perpendicular to the table so that the bottom edge swings to the left. The book is now facing away from you, perpendicular to the table resting on its spine. Now return the book to its original position. This time first rotate it 90° about a vertical axis, perpendicular to the table, so that the bottom swings to the left, then 90° about the axis that extends away from your body. It is now standing on its bottom edge and facing to the left. The order of the rotations was reversed and the end positions are clearly different. The manner in which rotations in three dimensions interact is exactly the way in which quaternions combine when multiplied, therefore quaternions are an excellent means of modeling rotations in space.

The Definition of Quaternions

A quaternion is often called a hypercomplex number because it is composed of four numbers. In parallel with complex numbers a quaternion may be expressed as the sum of four different components.

$$\mathbf{Q} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

As with complex numbers the $\mathbf{1}$ is unity in the real numbers and the \mathbf{i} , \mathbf{j} , and \mathbf{k} are unity in three different imaginary number systems. As with complex numbers we generally do not write the real unity.

The addition of quaternions is like complex numbers in that the coefficients of like components are added together.

$$\text{if } \mathbf{Q}_1 = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k} \text{ and } \mathbf{Q}_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k},$$

then

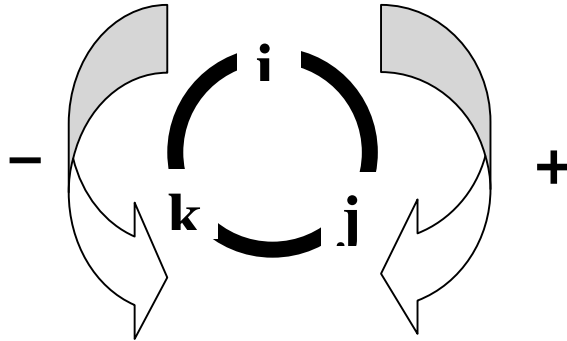
$$\mathbf{Q}_{12} = \mathbf{Q}_1 + \mathbf{Q}_2 = \mathbf{Q}_2 + \mathbf{Q}_1 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\mathbf{k}$$

Introduction to Quaternion Analysis

The multiplication of quaternions is like algebraic multiplication with one important exception, one must keep track of the order of the products of the components. First let us define the products of the components.

$$\begin{aligned} \mathbf{i} * \mathbf{j} &= -(\mathbf{j} * \mathbf{i}) = \mathbf{k} & \mathbf{j} * \mathbf{k} &= -(\mathbf{k} * \mathbf{j}) = \mathbf{i} & \mathbf{k} * \mathbf{i} &= -(\mathbf{i} * \mathbf{k}) = \mathbf{j} \\ \mathbf{i} * \mathbf{i} &= \mathbf{j} * \mathbf{j} = \mathbf{k} * \mathbf{k} = -\mathbf{1} \\ \mathbf{1} * \mathbf{i} &= \mathbf{i} & \mathbf{1} * \mathbf{j} &= \mathbf{j} & \mathbf{1} * \mathbf{k} &= \mathbf{k} \end{aligned}$$

From these relations it is clear why we say that there are three different imaginary numbers. Each is the $\sqrt{-1}$, but the product of any two is the third. To remember the relations in the first line the following diagram may be helpful. If one multiplies any two components in the clockwise order around the ring then the result is the third element. If they are multiplied in counter-clockwise order then the result is the negative of the third.



Now if we consider the product of two quaternions. Here we encounter one of the most difficult aspects of quaternion multiplication by hand. One must be very careful to retain the order of the products as one does the multiplication. Now that it is possible to write computer programs with subroutines to do quaternion multiplication it is not as much of a problem as it used to be.

$$\begin{aligned} \text{if } \mathcal{Q}_1 &= a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k} \text{ and } \mathcal{Q}_2 = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}, \text{ then} \\ \mathcal{Q}_1 * \mathcal{Q}_2 &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i} \\ &\quad + (a_1c_2 + c_1a_2 - b_1d_2 + d_1b_2)\mathbf{j} + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\mathbf{k} \end{aligned}$$

It may be proven that the product or sum of any two quaternions is another quaternion. Quaternions are also closed under subtraction and division.

The Trigonometric Form of a Quaternion

As with the complex numbers, there is a way to write a quaternion in a trigonometric format, which is often more convenient for analysis or understanding what is happening in a particular situation. We start with the magnitude of the quaternion, often called its tensor. The tensor of a quaternion is the square root of the sum of the squares of its components.

$$T = \sqrt{a^2 + b^2 + c^2 + d^2}$$

We can now express the quaternion as a real number, T , times a quaternion of magnitude 1.0, which will be called a unit quaternion. We note that the quaternion is made up of two types of entities, the real number, a , and a vector, $\mathbf{v} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. If we divide the quaternion by its magnitude we obtain a unit quaternion, \bar{Q} , that can be written as a trigonometric expression.

$$\begin{aligned} \mathbf{Q} &= a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \\ &= T * \bar{Q} = T * (\bar{a} + \bar{b}\mathbf{i} + \bar{c}\mathbf{j} + \bar{d}\mathbf{k}) \\ &= T(\bar{a} + \mathbf{v}) = T(\cos\phi + \sin\phi * \bar{\mathbf{v}}), \text{ where} \\ \cos\phi &= \frac{a}{\sqrt{a^2 + b^2 + c^2 + d^2}}, \quad \sin\phi = \frac{\sqrt{b^2 + c^2 + d^2}}{\sqrt{a^2 + b^2 + c^2 + d^2}}, \text{ and } \bar{\mathbf{v}} = \frac{b\mathbf{i} + c\mathbf{j} + d\mathbf{k}}{\sqrt{b^2 + c^2 + d^2}} \end{aligned}$$

The unit vector $\bar{\mathbf{v}}$ has a magnitude of 1.0. The vector \mathbf{v} will be called the vector of the quaternion. The angle ϕ will be called the angle of the quaternion.

Now we are at the point that we can connect the quaternion up with rotations in three-dimensional space. Before we start, it should be noted that though we started out looking for an extrapolation of the complex numbers to three dimensions in order to extend the concept of rotations, the interpretation is different for quaternions.

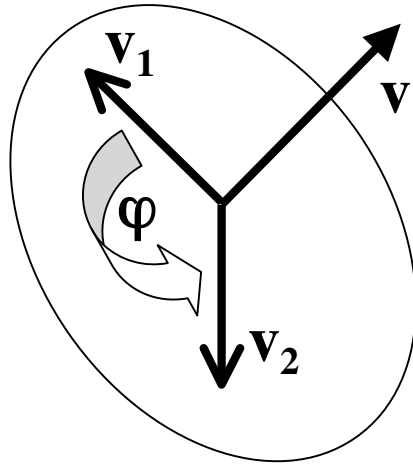
Rotations and Quaternions

We start with two arbitrary vectors in three-dimensional space. We wish to determine the ratio of the two vectors. In a sense we are dividing one vector by the other, an operation that has no meaning in standard vector analysis. First, we can translate the vectors so that they arise from a common point. This is not essential, since the vector's locations are irrelevant to their ratio, but it helps to visualize the process if they are so aligned. We now note that the two vectors

Introduction to Quaternion Analysis

define a plane that contains both. Within that plane, we can visualize rotating one vector through a particular angle, φ , to bring it into alignment with the other. Now the two vectors differ only in that one is a scalar multiple of the other. If the scaling factor is T and the unit vector, \mathbf{v} , is perpendicular to the plane of the two vectors and a rotation through an angle φ about that vector carries the first vector, \mathbf{v}_1 , into the second vector, \mathbf{v}_2 , then the ratio of the second vector to the first is the quaternion \mathcal{Q} .

$$\frac{\mathbf{v}_2}{\mathbf{v}_1} = \mathcal{Q} = T(\cos \varphi + \sin \varphi * \mathbf{v}) \Leftrightarrow \mathbf{v}_2 = \mathcal{Q} * \mathbf{v}_1$$



There is a caution that should be given at this time. As noted above, the order of multiplication is critical, therefore, it is relevant whether the ratio is written as

$$\mathbf{v}_2 * \mathbf{v}_1^{-1} = \mathcal{Q}_{21} \quad \text{or} \quad \mathbf{v}_1^{-1} * \mathbf{v}_2 = \mathcal{Q}_{12}.$$

You can choose either way, but must be consistent from then on.

If we call the two vectors α and β , then

$$\beta / \alpha = \beta * \alpha^{-1} = \mathbf{q} \quad \neq \quad \mathbf{q}' = \alpha^{-1} * \beta,$$

but

$$\alpha * \mathbf{q}' = \beta.$$

This is because the order of multiplication is relevant in quaternion analysis. In the expression

$$\alpha * \mathbf{q}' = \alpha * \alpha^{-1} * \beta$$

the alpha terms cancel, but they do not in the expression

Introduction to Quaternion Analysis

$$\mathbf{q}' * \alpha = \alpha^{-1} * \beta * \alpha .$$

A great deal more care and precision are required in quaternion analysis than with vector analysis, but the quaternion analysis is considerably more powerful. We will generally use the $\mathbf{q} = \beta * \alpha^{-1}$ form of the rotation quaternion.

A Simple Example:

Let α be \mathbf{i} and β be \mathbf{j} , then α^{-1} is $-\mathbf{i}$. If we go through the calculation as above –

$$\begin{aligned} \mathbf{q} &= \beta * \alpha^{-1} = \mathbf{j} * -\mathbf{i} = \mathbf{k} \\ \mathbf{q}' &= \alpha^{-1} * \beta = -\mathbf{i} * \mathbf{j} = -\mathbf{k} \end{aligned}$$

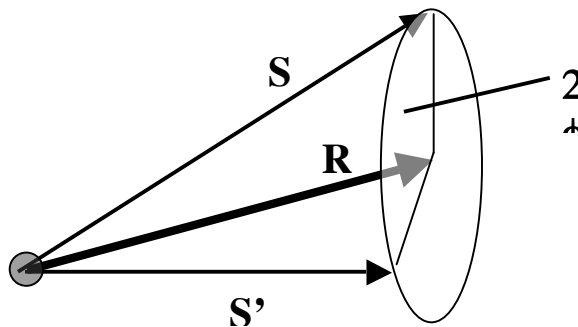
and thus

$$\begin{aligned} \mathbf{q} * \alpha &= \mathbf{k} * \mathbf{i} = \mathbf{j} = \beta \\ \alpha * \mathbf{q}' &= \mathbf{i} * -\mathbf{k} = \mathbf{j} = \beta . \end{aligned}$$

Therefore, the calculation works equally well with either interpretation of $\frac{\beta}{\alpha}$, but one has to be consistent and careful to observe the correct ordering of the products throughout.

Euler's Equation

The arguments introduced in the previous section show that quaternions may be used to model rotations of vectors in a plane, however, most rotations do not occur in a plane. Usually the rotated vector sweeps out a cone in space. These types of rotations are modeled by a more complicated expression, but still using quaternions.



Introduction to Quaternion Analysis

Let the rotating vector be called \mathbf{S} and suppose that it is being swept in a conical path about the unit vector \mathbf{v} through an angle 2ϕ . We write out the rotation quaternion that has the quaternion angle ϕ and the unit vector \mathbf{v} as follows –

$$\mathbf{R} = \cos \phi + \sin \phi * \mathbf{v};$$
$$\mathbf{v} = \frac{\mathbf{R}}{|\mathbf{R}|}; \quad \text{where } |\mathbf{R}| \text{ is the magnitude of } \mathbf{R}.$$

It can be shown that the vector \mathbf{S}' , which represents the rotated \mathbf{S} vector, is given by the expression

$$\mathbf{S}' = \mathbf{R} * \mathbf{S} * \mathbf{R}^{-1}$$

which is known as Euler's equation.

The inverse of a quaternion is given by the definition -

$$\mathbf{R}^{-1} = \frac{1}{\mathbf{R}} = \frac{1}{T^2} * [\cos \phi - \sin \phi * \mathbf{v}] .$$

Since we are using unit quaternions for rotations the tensor is unity, therefore the tensor of its inverse is also unity. Note that the angle of the rotation quaternion is half of the angular excursion that the rotating vector experiences.

Euler's equation is central to all that follows. Virtually all rotations of moving objects are conical rotations, therefore may be modeled as quaternion multiplication according to Euler's equation. If the rotation occurs entirely in a single plane one may use simple quaternion multiplication. If it is conical, then one must use Euler's equation.

Orientation

Quaternions give us a conceptual tool to compute the consequences of rotations, but a little experimentation will show that they are not sufficient. The other tool we need is a mechanism for coding orientation. It is obvious that anatomical structures have a location and an extension in space. If one is considering the movements of the hand, it clearly moves from one location to another and it occupies a finite volume, which can be expressed as its length, width, thickness or some combination of those. In addition to these properties it is orientable, meaning that it has a top and bottom, a medial and a lateral aspect, and distal and proximal parts. Generally, it is not

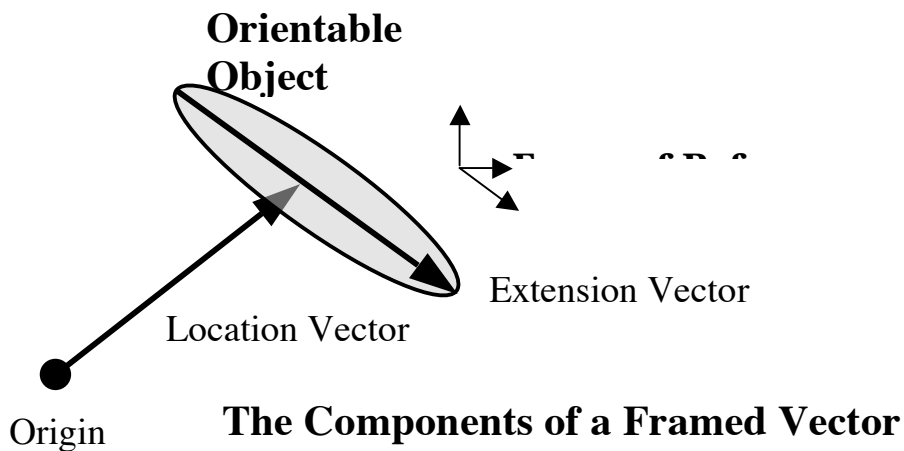
Introduction to Quaternion Analysis

sufficient to place your hand in a particular location, you must also have it properly oriented if you wish to hold or catch an object or make a gesture. What we need is a way of expressing the orientation in a way that allows us to manipulate it mathematically. If we are able to express orientation in a form that can be quantitatively manipulated, then it will be possible to calculate the changes in orientation that occur when an anatomical object is rotated.

The way that orientation will be codified is to determine three non-coplanar vectors that align with certain aspects of the anatomical object. For instance, in specifying the orientation of a hand, one might choose a vector that points in the direction of the middle finger, one perpendicular to the first in the direction of the thumb, and a third perpendicular to the first two vectors and extending in the direction of the hand's dorsal surface. While there is no reason that the three vectors need to be mutually perpendicular, it is often convenient to select them that way. It makes it easier to visualize the transformations that occur with movements.

Framed Vectors

We will call such an array of vectors a frame of reference, or often just a frame. Ultimately we will combine the three vectors in the frame of reference with a vector that indicates the location of an anatomical object and a vector that indicates its extension and call the whole array of five vectors a framed vector. Note that while we will consider the frame of reference as a component of a framed vector, it is not attached to the other two vectors.



Introduction to Quaternion Analysis

When an orientable structure rotates all the components of the frame of reference rotate in the same manner, so that we can multiply each component by the same rotation quaternion, using Euler's Formula. For every rotation there will be changes in at least two of the frame's components and usually in all three.

Variations on the Concept of a Framed Vector

While we define a framed vector to have a location vector, an extension vector, and a frame of reference that is a set of three non-coplanar vectors, there is no reason that one can not work with variations on that basic structure. In some instances it may be expeditious to have more than one extension vector. In many situations we are really concerned only with the changes in orientation and will therefore work only with the frame of reference. Many times we will have one of the frame of reference vectors aligned with the extension vector, therefore it may be redundant to calculate the same transformation twice. Frequently the location of the orientable object is taken to be the origin of the coordinate system, therefore the location vector is null.

When dealing with complex anatomical objects one may utilize arrays of framed vectors that express structure of various components of the object. For instance to describe the movements of a spinal vertebra one might have framed vectors for the superior and inferior surfaces of the vertebral body, each facet joint and the vertebral spine. Such an array of framed vectors will be called a box, because it contains the modeled object in the sense of defining a space that is congruent with the features of concern. In this essay, we will deal only with single framed vectors or frames of reference.

Changes in the Components of Framed Vectors with Movements

There are three basic types of movements that may occur; translation, rotation, and rescaling. Translation is a straight line movement during which the object does not revolve in any direction. Rotation is a movement in which the object and its parts move about an axis of rotation. Rescaling is a change in the size of the object so that it expands or contracts in one or more directions.

Frames of reference are not changed by translation or changing magnitude, but they are always changed by finite rotations. An object's extension is not changed by translation, but it is

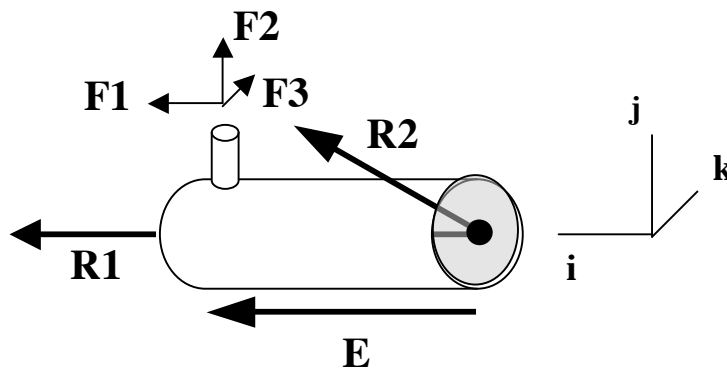
Introduction to Quaternion Analysis

always changed by rescaling and usually by rotation. The location of the object may or may not be changed by rescaling or rotation, but is always changed by finite translations. It usually would not be changed by rescaling. It would stay the same if the rotation occurred about the head of the location vector, but not otherwise. A single movement may be a combination of any pair of movement types or all three combined.

Types of Rotation

Rotation is conveniently differentiated into several types, though the different types are really just different parts of a continuous spectrum of rotations. If the rotation occurs about the axis of a vector, then we say that the vector experiences a pure spin about that vector. The vector is not changed by a pure spin. If the axis of rotation is perpendicular to the direction of a vector, then the vector is said to experience a pure swing about that vector. The vector stays in a plane that is perpendicular to the axis of rotation. Such transformations may be expressed by quaternion multiplication. A rotation that occurs about an axis that lies at any angle other than 0° or 90° to the direction of a vector will cause the vector to experience a conical rotation, often simply referred to as a swing. Conical rotations may be expressed as Euler products. Using Euler's formula will always give the correct transformation of a rotated vector, therefore, if there is any doubt about the relative directions of the rotation quaternion's vector and the direction of the rotating vector it is prudent to use Euler's formula.

An Example



Up to this point the descriptions have been fairly qualitative. Let us now consider an example in which we apply these concepts to a problem. The situation is illustrated above. There is a

Introduction to Quaternion Analysis

cylinder that lies with its long axis along the \mathbf{i} axis. Sticking out from that cylinder is a small peg that is aligned with the \mathbf{j} axis. Initially, we will consider the case where the cylinder rotates about an axis parallel with its long axis and through the center of the cylinder, \mathbf{R}_1 ($R1$ in the diagram). The location vector is set to null for the present calculation, meaning that we assume the origin of the system lies along the axis of rotation. The extension vector, \mathbf{E} , is taken to be a vector that lies parallel with the long axis of the cylinder with a length equal to the length of the cylinder. An orientation frame is chosen so that the first component ($\mathbf{F1}$) is parallel with the extension vector, or the long axis of the cylinder. The second component ($\mathbf{F2}$) is aligned with the long axis of the peg and the third component ($\mathbf{F3}$) is taken to be perpendicular to the other two and directed so that if the fingers of your right hand sweep from the first to the second component then your thumb points in the direction of the third component. This is called a right-handed coordinate system. As is usual, it is assumed that all the orientation frame's vector are unit vectors. If we write all this out in a matrix of vectors then the framed vector, \mathbf{F} , would be as follows.

$$\mathbf{F} = \begin{Bmatrix} \mathbf{L} \\ \mathbf{E} \\ \mathbf{F1} \\ \mathbf{F2} \\ \mathbf{F3} \end{Bmatrix} = \begin{Bmatrix} 0 & 0 & 0 \\ l * \mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 \\ 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{k} \end{Bmatrix}$$

The rotation quaternion is given by the expression

$$\mathbf{R}_1 = \cos \frac{\theta}{2} + \mathbf{i} * \sin \frac{\theta}{2}, \text{ where } \theta \text{ is the angular excursion}$$

Consequently, after rotation through the angle θ , the framed vector, \mathbf{F}' , is given by the following expression.

$$\mathbf{F}' = \mathbf{R}_1 * \begin{Bmatrix} \mathbf{L} \\ \mathbf{E} \\ \mathbf{F1} \\ \mathbf{F2} \\ \mathbf{F3} \end{Bmatrix} * \mathbf{R}_1^{-1} = \left(\cos \frac{\theta}{2} + \mathbf{i} * \sin \frac{\theta}{2} \right) * \begin{Bmatrix} 0 & 0 & 0 \\ l * \mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 \\ 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{k} \end{Bmatrix} * \left(\cos \frac{\theta}{2} - \mathbf{i} * \sin \frac{\theta}{2} \right)$$

Introduction to Quaternion Analysis

$$\mathbf{F}' = \begin{Bmatrix} \mathbf{L}' \\ \mathbf{E}' \\ \mathbf{F1}' \\ \mathbf{F2}' \\ \mathbf{F3}' \end{Bmatrix} = \begin{Bmatrix} 0 & 0 & 0 \\ \mathbf{l} * \mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 \\ 0 & \mathbf{j} * \cos \theta & \mathbf{k} * \sin \theta \\ 0 & -\mathbf{j} \sin \theta & \mathbf{k} * \cos \theta \end{Bmatrix}$$

Looking at the expression, it is clear that the rotation did not change the vectors that were parallel with the axis of rotation, those in the first column. The vectors in the perpendicular directions were transformed in the manner that one would expect if they rotated through the angle θ about the \mathbf{i} axis. If we plug a 90° rotation in the expression, then the $\mathbf{F2}$ axis is directed in the direction of the \mathbf{k} axis and the $\mathbf{F3}$ axis is directed in the direction of the negative \mathbf{j} axis.

These results are basically what we would expect if we visualize the rotation occurring as described. With some thought we could have written down the results without performing the calculations. However, the situation is a great deal more difficult if we have to determine the result of rotation about an axis that is oblique to the long axis of the cylinder. It is in these more difficult situations that the benefit of the quaternion approach becomes evident. In the following section we will perform the calculation using quaternions and orientation expressed in a framed vector.

A More Difficult Example

We start with the axis of rotation, which is a unit vector in the direction of $\mathbf{i} + \mathbf{j}$, and the angular excursion is θ , therefore the angle used in the Euler formula is $\theta/2$. The framed vector is the same as in the previous example.

$$\mathbf{R1} * \begin{Bmatrix} \mathbf{L} \\ \mathbf{E} \\ \mathbf{F1} \\ \mathbf{F2} \\ \mathbf{F3} \end{Bmatrix} * \mathbf{R1}^{-1} = \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \left(\frac{\mathbf{i}}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}} \right) \right) * \begin{Bmatrix} 0 & 0 & 0 \\ \mathbf{l} * \mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 \\ 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{k} \end{Bmatrix} * \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \left(\frac{\mathbf{i}}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}} \right) \right)$$

$$= \left(\cos\varphi + \frac{1}{\sqrt{2}} \sin\varphi * (\mathbf{i} + \mathbf{j}) \right) * \begin{Bmatrix} 0 & 0 & 0 \\ l * \mathbf{i} & 0 & 0 \\ 0 & \mathbf{j} & 0 \\ 0 & 0 & \mathbf{k} \end{Bmatrix} * \left(\cos\varphi - \frac{1}{\sqrt{2}} \sin\varphi * (\mathbf{i} + \mathbf{j}) \right)$$

The calculation is not presented here, but it is straightforward and can be easily carried out by following the standard rules of quaternion multiplication.

$$\begin{Bmatrix} \overline{\mathbf{L}'} \\ \overline{\mathbf{E}'} \\ \mathbf{F1}' \\ \mathbf{F2}' \\ \mathbf{F3}' \end{Bmatrix} = \begin{Bmatrix} \frac{0}{\frac{l * \mathbf{i} * \cos^2 \varphi}{2}} & \frac{0}{\frac{l * \mathbf{j} * \sin^2 \varphi}{2}} & \frac{0}{l * -\mathbf{k} * \sqrt{2} * \cos \varphi \sin \varphi} \\ \frac{\mathbf{i} * \cos^2 \varphi}{2} & \frac{\mathbf{j} * \sin^2 \varphi}{2} & -\mathbf{k} * \sqrt{2} * \cos \varphi \sin \varphi \\ \frac{\mathbf{i} * \sin^2 \varphi}{2} & \frac{\mathbf{j} * \cos^2 \varphi}{2} & \mathbf{k} * \sqrt{2} * \cos \varphi \sin \varphi \\ \mathbf{i} * \sqrt{2} * \cos \varphi \sin \varphi & -\mathbf{j} * \sqrt{2} * \cos \varphi \sin \varphi & \mathbf{k} * (\cos^2 \varphi - \sin^2 \varphi) \end{Bmatrix}$$

$$\begin{Bmatrix} \overline{\mathbf{L}'} \\ \overline{\mathbf{E}'} \\ \mathbf{F1}' \\ \mathbf{F2}' \\ \mathbf{F3}' \end{Bmatrix} = \begin{Bmatrix} \frac{0}{\frac{l * \mathbf{i} * (1 + \cos \theta)}{2}} & \frac{0}{\frac{l * \mathbf{j} * (1 - \cos \theta)}{2}} & \frac{0}{-\mathbf{k} * \frac{l}{\sqrt{2}} * \sin \theta} \\ \frac{\mathbf{i} * (1 + \cos \theta)}{2} & \frac{\mathbf{j} * (1 - \cos \theta)}{2} & -\mathbf{k} * \frac{1}{\sqrt{2}} * \sin \theta \\ \frac{\mathbf{i} * (1 - \cos \theta)}{2} & \frac{\mathbf{j} * (1 + \cos \theta)}{2} & \mathbf{k} * \frac{1}{\sqrt{2}} * \sin \theta \\ \mathbf{i} * \sin \theta & -\mathbf{j} * \sin \theta & \mathbf{k} * \cos \theta \end{Bmatrix}$$

At this point we can plug in a value of $\theta = \pi$ radians = 180° and the framed vector is given by the following evaluation.

$$\begin{Bmatrix} \overline{\mathbf{L}'} \\ \overline{\mathbf{E}'} \\ \mathbf{F1}' \\ \mathbf{F2}' \\ \mathbf{F3}' \end{Bmatrix} = \begin{Bmatrix} \frac{0}{\frac{l * \mathbf{i} * (1 - 1)}{2}} & \frac{0}{\frac{l * \mathbf{j} * (1 + 1)}{2}} & \frac{0}{-\mathbf{k} * \frac{l}{\sqrt{2}} * 0} \\ \frac{\mathbf{i} * (1 - 1)}{2} & \frac{\mathbf{j} * (1 + 1)}{2} & -\mathbf{k} * \frac{1}{\sqrt{2}} * 0 \\ \frac{\mathbf{i} * (1 + 1)}{2} & \frac{\mathbf{j} * (1 - 1)}{2} & \mathbf{k} * \frac{1}{\sqrt{2}} * 0 \\ \mathbf{i} * 0 & -\mathbf{j} * 0 & \mathbf{k} * -1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & 0 & 0 \\ 0 & l * \mathbf{j} & 0 \\ 0 & \mathbf{j} & *0 \\ \mathbf{i} & 0 & 0 \\ 0 & 0 & -\mathbf{k} \end{Bmatrix}$$

Introduction to Quaternion Analysis

If you are good at visualization, you will be able to see that this is the correct framed vector. Otherwise, the quaternion multiplication is a much easier way to determine the consequences of such a rotation.

If we ask what the framed vector would be for a 90° rotation, then even good visualization skills are severely challenged.

$$\begin{aligned}
 \begin{Bmatrix} \mathbf{L}' \\ \mathbf{E}' \\ \mathbf{F1}' \\ \mathbf{F2}' \\ \mathbf{F3}' \end{Bmatrix} &= \begin{Bmatrix} \frac{0}{l * i * \frac{(1+0)}{2}} & \frac{0}{l * j * \frac{(1-0)}{2}} & \frac{0}{-k * \frac{l}{\sqrt{2}} * 1} \\ \frac{i * \frac{(1+0)}{2}}{2} & \frac{j * \frac{(1-0)}{2}}{2} & \frac{-k * \frac{1}{\sqrt{2}} * 1}{\sqrt{2}} \\ \frac{i * \frac{(1-0)}{2}}{2} & \frac{j * \frac{(1+0)}{2}}{2} & \frac{k * \frac{1}{\sqrt{2}} * 1}{\sqrt{2}} \\ i * 1 & -j * 1 & k * 0 \end{Bmatrix} \\
 &= \begin{Bmatrix} \frac{0}{\frac{l * i}{2}} & \frac{0}{\frac{l * j}{2}} & \frac{0}{\frac{-l * k}{\sqrt{2}}} \\ \frac{i}{2} & \frac{j}{2} & \frac{-k}{\sqrt{2}} \\ \frac{i}{2} & \frac{j}{2} & \frac{k}{\sqrt{2}} \\ i & -j & 0 \end{Bmatrix}
 \end{aligned}$$

For any other angle of rotation, it would be virtually impossible to write down the rotated framed vector without computation. The value of this approach is in those situations where the angle is not a convenient fraction of a circle. In real problems, the rotations are almost never a convenient angular excursion.

Derivation of the Rotation Quaternion Given the Initial and Final Frames of Reference

Up to this point we have started with the initial orientation and the rotation quaternion and have computed the final orientation. Another situation that frequently arises is that we have obtained the initial and final orientations and we need to determine a single rotation that will transform the initial frame of reference into the final frame of reference. We will now consider how that is accomplished.

Introduction to Quaternion Analysis

First let us set up the problem by actually working it forward and then trying to work it backwards. Let the frame of reference be aligned with the axes of a right-handed coordinate system.

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k} \end{bmatrix}$$

Let the rotation be about the direction of $\mathbf{v} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$, so the rotation quaternion is given by the quaternion \mathbf{R} . Its half angle form is given by \mathbf{r} .

$$\begin{aligned} \mathbf{R} &= \cos \theta + \sin \theta * \mathbf{v} \\ \mathbf{r} &= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} * \mathbf{v} = \cos \phi + \sin \phi * \mathbf{v} \end{aligned}$$

We can compute the transformation of the frame from the initial orientation to the final orientation as follows.

$$\begin{aligned} \mathbf{F}' &= \left(\cos \phi + \sin \phi * \frac{(\mathbf{i} + \mathbf{j} + \mathbf{k})}{\sqrt{3}} \right) * \begin{bmatrix} \mathbf{i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k} \end{bmatrix} * \left(\cos \phi - \sin \phi * \frac{(\mathbf{i} + \mathbf{j} + \mathbf{k})}{\sqrt{3}} \right) \\ \mathbf{F}' &= \begin{bmatrix} \left(\frac{\cos \theta - 1}{6} + \frac{1 + \cos \theta}{2} \right) * \mathbf{i} & \left(\frac{1 - \cos \theta}{3} + \frac{\sin \theta}{\sqrt{3}} \right) * \mathbf{j} & \left(\frac{1 - \cos \theta}{3} - \frac{\sin \theta}{\sqrt{3}} \right) * \mathbf{k} \\ \left(\frac{1 - \cos \theta}{3} - \frac{\sin \theta}{\sqrt{3}} \right) * \mathbf{i} & \left(\frac{\cos \theta - 1}{6} + \frac{1 + \cos \theta}{2} \right) * \mathbf{j} & \left(\frac{1 - \cos \theta}{3} + \frac{\sin \theta}{\sqrt{3}} \right) * \mathbf{k} \\ \left(\frac{1 - \cos \theta}{3} + \frac{\sin \theta}{\sqrt{3}} \right) * \mathbf{i} & \left(\frac{1 - \cos \theta}{3} - \frac{\sin \theta}{\sqrt{3}} \right) * \mathbf{j} & \left(\frac{\cos \theta - 1}{6} + \frac{1 + \cos \theta}{2} \right) * \mathbf{k} \end{bmatrix} \\ &= \begin{bmatrix} \kappa_1 * \mathbf{i} & \kappa_2 * \mathbf{j} & \kappa_3 * \mathbf{k} \\ \kappa_3 * \mathbf{i} & \kappa_1 * \mathbf{j} & \kappa_2 * \mathbf{k} \\ \kappa_2 * \mathbf{i} & \kappa_3 * \mathbf{j} & \kappa_1 * \mathbf{k} \end{bmatrix} \end{aligned}$$

The array of solutions looks imposing when written out, but it is just three values that are shifted one position to the right in each successive row.

A little thought will lead one to the conclusion that a rotation of 120° will transform the frame from being aligned with the coordinate axes to being aligned again but shifted one axis clockwise.

Introduction to Quaternion Analysis

$$\sin 120^\circ = \frac{\sqrt{3}}{2}, \quad \cos 120^\circ = -\frac{1}{2}$$

$$\kappa_1 = 0.0, \quad \kappa_2 = 1.0, \quad \kappa_3 = 0.0$$

$$\mathbf{F}' = \begin{bmatrix} \mathbf{F}'_1 \\ \mathbf{F}'_2 \\ \mathbf{F}'_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k} \\ \mathbf{i} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Substituting 120° for θ in the expression confirms that outcome.

We now know the actual rotation quaternion for the transformation of the orientation's frame of reference. To work backwards from the two frames of reference to the rotation quaternion involves finding two quaternion multiplication's that when combined will give the correct rotation quaternion. We start by finding the quaternion that will transform the first axis of the orientation frame before rotation into the same axis after rotation, \mathbf{F}_1 into \mathbf{F}'_1 . That is accomplished by taking the ratio of the final to the initial axes.

$$\begin{aligned} R_{s w} &= \frac{\mathbf{F}'_1}{\mathbf{F}_1} = \mathbf{F}'_1 * \mathbf{F}_1^{-1} \\ &= \mathbf{j} * -\mathbf{i} = \mathbf{k} \end{aligned}$$

We could pick any axis to calculate the first ratio.

Once we have the rotation quaternion, we need to calculate an intermediate frame of reference by multiplying the other two axes by the same quaternion, but whereas $R_{s w}$ will transform \mathbf{F}_1 into \mathbf{F}'_1 by quaternion multiplication we need to use Euler's formula to transform the other two axes.

$$\begin{aligned} \mathbf{F}_1 &= \left(\frac{1}{\sqrt{2}} + \frac{\mathbf{k}}{\sqrt{2}} \right) * \begin{bmatrix} \mathbf{i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k} \end{bmatrix} * \left(\frac{1}{\sqrt{2}} - \frac{\mathbf{k}}{\sqrt{2}} \right) \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{j} & \mathbf{0} \\ -\mathbf{i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k} \end{bmatrix} \end{aligned}$$

Introduction to Quaternion Analysis

We note that the second and third frame vectors in the intermediate frame are not the same as in the final frame, therefore we divide the final \mathbf{F}_2 by the intermediate \mathbf{F}'_1 to obtain the quaternion that will rotate the frame of reference from the intermediate orientation to the final orientation.

$$\begin{aligned} R_{S P} &= \frac{\mathbf{F}'_2}{\mathbf{F}_{2:i}} = \mathbf{F}'_2 * \mathbf{F}_{2:i}^{-1} \\ &= \mathbf{k} * \mathbf{i} = \mathbf{j} \end{aligned}$$

Then we have to multiply the intermediate frame by this second rotation quaternion to obtain the final frame of reference.

$$\begin{aligned} \mathbf{F}_F &= \left(\frac{1}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}} \right) * \begin{bmatrix} \mathbf{0} & \mathbf{j} & \mathbf{0} \\ -\mathbf{i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k} \end{bmatrix} * \left(\frac{1}{\sqrt{2}} - \frac{\mathbf{j}}{\sqrt{2}} \right) \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k} \\ \mathbf{i} & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{aligned}$$

Note that the movement spins the frame of reference about the axis that we obtained in the first calculation. This will always be the case.

Finally, the rotation quaternion that performs the same transformation as these two rotation quaternions taken successively is the product of the two quaternions.

$$\begin{aligned} \mathbf{R} &= R_{S P} * R_{S W} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}} \right) * \left(\frac{1}{\sqrt{2}} + \frac{\mathbf{k}}{\sqrt{2}} \right) \\ &= \frac{1}{2} * (1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \end{aligned}$$

If we look at the scalar term, the real term, then we can determine the angle of the quaternion.

$$\cos \gamma = \frac{1}{2} \Leftrightarrow \gamma = \frac{\pi}{3} = 60^\circ$$

Since we are using Euler's formula we have to double the angle of excursion to obtain the angle of the quaternion, therefore the angle of the quaternion is 120° and the unit vector of the quaternion is obtained as follows.

Introduction to Quaternion Analysis

$$\begin{aligned} \mathbf{R} &= \frac{1}{2} + \frac{1}{2} * (\mathbf{i} + \mathbf{j} + \mathbf{k}) = \cos \gamma + \sin \gamma * \mathbf{v} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2} * \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} = \cos \frac{\pi}{3} + \sin \frac{\pi}{3} * \mathbf{v} \\ \mathbf{v} &= \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} \end{aligned}$$

We have solved a particular case in order to demonstrate that the method works and so that it is easier to see how it works, but the same approach will always yield a solution if it is possible to find one.

Note that we could chose any two frame axes to obtain the result. In some instances it might be more convenient to choose a different set. The intermediate results will be different, but the product will always be the same for any particular situation.

In practice, this ability to determine the rotation that would bring about a particular transformation of the orientation is often one of the most useful tools of quaternion analysis of systems of orientable objects. It is possible to show that the two intermediate rotation vectors that are ratios of the frame's axes are quantitative expressions of the swing and spin that turn up frequently is discussions of concurrent rotations. They were labeled in the derivation that we just completed by the subscripts $\mathbf{R}_{\mathbf{S W}}$ and $\mathbf{R}_{\mathbf{S P}}$, for swing and spin, respectively. We will not address those points here, but they are treated elsewhere is considerable detail

Summation

I have tried to introduce the basic concepts that lie at the foundations of the quaternion analysis of the movements of orientable objects. There are remarkably few concepts required, but there is a subtlety about them that often leads to very elegant results with comparatively little manipulation. . Very little more is required than what has been presented here. Most of the analysis lies in applying these concepts to specific problems and carrying out the calculations for a variety of situations. The basic concepts are straightforward and intuitive, they readily yield to quantitative analysis, and make it possible to ask very detailed questions about an anatomical system and to obtain very precise results. At the same time the symbolism is such that one can reason at a fairly high level of abstraction and still obtain specific results.

Introduction to Quaternion Analysis

It should be stressed that this essay only skims the surface of the many facets of quaternion analysis of moving systems. There are many interesting problems in this area and a fair number have already been analyzed to a greater or lesser extent. Many of these are treated in other essays.

It will be apparent to anyone who tries to carry out the calculations that were not written out in full that there is a fair amount of drudgery in doing the actual calculations and a great deal of care must be taken to maintain the order of multiplication. However, this is also a drawback of methods based on matrices. It is probably implicit in any adequate system for quantitative analysis for such systems. When dealing with matrices one normally uses a computer to handle the actual calculations. Similarly, if one does a great many calculations with quaternions it is prudent to avail oneself of a computer program that allows the automatic computation of the quaternion products. Such programs are easily written.