Movements of Orientable Objects:

The Application of Quaternions to the Analysis of Movement in Three Dimensional Space

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Mathematical Objects Used to Model Movements of Orientable Objects

Three types of mathematical objects will be used to model or describe the movements of orientable objects. These are scalars, vectors, and quaternions. Scalars are single real or complex numbers, which can represent magnitudes, such as, length, area, or temperature. Vectors can represent physical entities that have both magnitude and direction, such as displacements, velocities, accelerations, momenta, and forces. One may characterize a vector by an ordered one dimensional array of real or complex numbers. Quaternions are a combination of scalars and vectors, generally a linear sum of a scalar and a vector. They can be used to represent orientable objects, their movements in threedimensional space, and, eventually, combinations of linear and rotatory forces, called wrenches because they may are a twisting force couple combined with a simultaneous linear pushing or pulling force.

To indicate the type of object being considered in an expression, the following conventions will be adopted. Scalars will be represented by italicized symbols, such as - a, x, θ, or T. Vectors will be represented by bold symbols, such as - $\mathbf{a}$, $\mathbf{x}$, $\boldsymbol{\theta}$, and $\mathbf{T}$. Quaternions will be represented by symbols that are both bold and italicized, such as - $\boldsymbol{a}$, $\boldsymbol{x}$, $\boldsymbol{\theta}$, and $\boldsymbol{T}$. While numbers are generally scalars, they will be plain text unless they represent a physical quantity.

In addition, different sets of symbols will generally be used for the different types of mathematical objects. For instance, $\boldsymbol{q}$ will generally represent a quaternion, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ will generally represent vectors, and a, b, and c will generally represent scalars. Also, in some expressions arrows may be used to signify that a variable is either a vector, $\vec{\mathbf{v}}$, or a quaternion, $\vec{\boldsymbol{r}}$. The reason for using the two sided arrow for vectors and the one-sided arrow for quaternions will be come evident as we go along.

Scalars

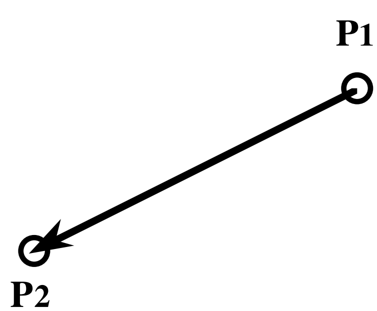
Scalars are mathematical objects that can be represented with real numbers. On occasion they might also be represented by a complex number, but complex numbers are more often considered to be a two dimenisonal vector. Scalars are used commonly enough that there is no need to spend much time on their definition and the operations that can be performed with them, since they obey the rules of ordinary arithmetic..

It may be relevant to pause momentarily and consider the fact that the set of real numbers contains of several different types of numbers. The smallest subset of real numbers is the set of positive integers. The positive integers are closed under addition and multiplication, meaning that if one adds a positive integer to another, the result is also a positive integer. Similarly, for the multiplication of positive integers. If we allow subtraction, then there are expressions that can not be evaluated with positive integers, such as $2 - 1 = \ ?$, which requires that we add the negative integers to the set of numbers. If we allow division, then there are expressions that can not be evaluated with integers, such as $1/3 = ?$, which require that we add the rational numbers to our set of numbers. Still there are expressions in rational numbers that can not be evaluated with rational numbers, such as $\sqrt{2}$, therefore we must greatly expand the set of numbers by adding the set of irrational numbers, to form the set of real numbers. We could go another step along this progression and show that there are simple expressions in real numbers that can not be evaluated with a real number, such as $\sqrt{-1}$, and expand our definition of number to include imaginary numbers forming the set of complex numbers. While we may use complex numbers as we go along, the scalars which are generally used will be restricted to the real numbers.

Complex numbers are relevant to our discussion from a historical point of view, because Hamilton first created quaternions as a generalization of the concept of complex numbers, therefore the attributes of the complex number must be a subset of the attributes of quaternions. There are certain properties of complex numbers that will be examined in some detail because they give insight into similar attributes of quaternions.

Vectors

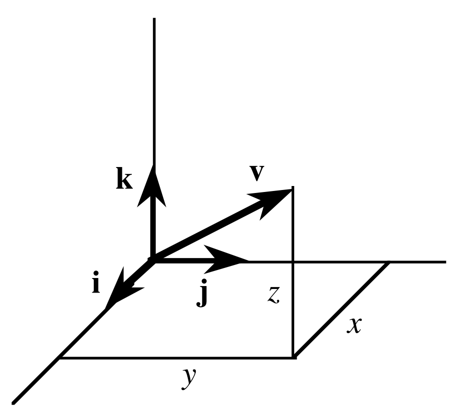
Complex numbers can, and often are, represented as vectors in a plane, but we will be primarily concerned with vectors in three dimensional spaces, where the analogy with complex numbers is not relevant. In fact, Hamilton's generalization of complex numbers, the quaternions, are very useful for the description of movements in three dimensional space. However, we will come to this point gradually, by considering the nature of vectors and showing how the concept of quaternions flows naturally from the multiplication of vectors.



Definition of a Vector: *A vector will be taken to be the difference between two points in space.* The difference between a point, $\mathbf{P}_1$, and another point, $\mathbf{P}_2$, will be the taken to be the shortest path from $\mathbf{P}_1$ to $\mathbf{P}_2$. $\mathbf{P}_1$ is the *origin* of the vector and $\mathbf{P}_2$ is the *terminus*. Since the shortest path from $\mathbf{P}_1$ to $\mathbf{P}_2$ has a definite direction, unless $\mathbf{P}_2$ is identical with $\mathbf{P}_1$, and a definite length, a vector has both magnitude and direction.

It is possible to generalize the concept of vectors to many dimensional, even infinite dimensional, spaces, but we will generally be concerned with three dimensional spaces, so, unless specified to be otherwise, the space under consideration will be considered to be a three dimensional.

Components of a Vector: In order to compute with vectors it is convenient to establish cardinal directions and magnitudes for measuring the length of vectors. This is generally done by introducing three independent standard vectors that are embedded in the space and that determine three non-coplanar directions and scales in each direction for measuring distance. The mathematics is simplest if these three cardinal vectors originate from a common origin and are equal in length and mutually perpendicular. We will assume such a set of cardinal vectors, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, for space. There are two fundamentally different ways that one can set up these vectors. If one curls the fingers of one's right hand so that the fingers extend from positive **i** to positive **j**, and the thumb points in the direction of positive **k**, then the system is said to be right-handed. If the positive **k** axis points in the direction opposite to the direction of the thumb, then the system is left-handed. There is no way that a right handed system can be converted into a left handed system using translations, rotations, or positive changes in scale. We will generally use right-handed coordinate systems for space, but there is no reason that a left-handed coordinate system would not work as well. There may be occasions when a left handed system would be more appropriate or convienient for analysis.



Any point, **P**, in space can be expressed as a function of the cardinal vectors as follows. Extend lines along the axes of the three cardinal vectors. Construct perpendicular lines from each of these axial lines to the point **P**. Measure the distance from the origin of the coordinate vectors to the intersections with the perpendicular lines by counting the number of times that one can lay the cardinal vector along its axial line from its origin to the point of intersection. That is the distance that the point, **P**, lies in that direction. Therefore the position of a point, **P**, can be specified by an ordered triplet of numbers, *{x, y, z}*, the distances in the directions of the three cardinal vectors. Or expressing the point **P** in terms of the coordinate vectors -

$$\mathbf{P} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,.$$

Points in Space as Displacement Vectors from the Origin: You may have noticed that the points in space that have been mentioned so far have been written as vectors. That is because they may be visualized as vector displacements from the origin of the coordinate system, since the origin of the coordinate system may be written as $\mathbf{O} = 0\mathbf{i} + 0\,\mathbf{j} + 0\mathbf{k}$ and the difference between two points is the distances between their projections upon the coordinate axes

$$\mathbf{P}_2 - \mathbf{P}_1 = \mathbf{v}_{12} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}\,,$$

it follows that -

$$\begin{aligned}\mathbf{P}_1 - \mathbf{O} &= (x_1 - 0)\mathbf{i} + (y_1 - 0)\,\mathbf{j} + (z_1 - 0)\mathbf{k} \\ &= x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k} = \mathbf{P}_1\end{aligned}\,.$$

Magnitude of Vectors: The length of the vector that passes from point $\mathbf{P}_1$ to point $\mathbf{P}_2$, $|\mathbf{v}_{12}|$, is the square root of the sum of the squares of the differences on each axis. Written symbolically -

$$\mathbf{P}_2 - \mathbf{P}_1 = \mathbf{v}_{12} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k} \quad \text{thus}$$

$$|\mathbf{P}_2 - \mathbf{P}_1| = |\mathbf{v}_{12}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,.$$

Unit Vectors: The direction of the vector from $\mathbf{P}_1$ to point $\mathbf{P}_2$, $\mathbf{U}[\mathbf{v}_{12}]$ is represented by a vector of length 1.0 in the same direction as $\mathbf{v}_{12}$. One could construct such a vector if one took the vector from $\mathbf{P}_1$ to $\mathbf{P}_2$ and divided it by its length. Symbolically -

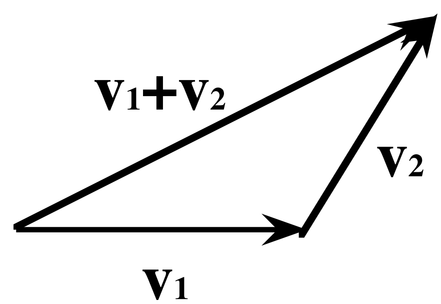
$$U[\mathbf{P}_2 - \mathbf{P}_1] = \frac{\mathbf{P}_2 - \mathbf{P}_1}{|\mathbf{P}_2 - \mathbf{P}_1|} = \frac{\mathbf{v}_{12}}{|\mathbf{v}_{12}|} = \frac{(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} = \mathbf{U}[\mathbf{v}_{12}]$$

Equality of Vectors: Two vectors are equal if and only if they have the same magnitude and direction. They do not necessarily have the same origin or terminus, but they must be parallel and the same length. A vector is defined to the extent that given its origin the terminus is specified and *vice versa*. In some situations, the origin is specified, therefore the vector is unique.

Addition of Vectors: The sum of two vectors $\mathbf{v}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and $\mathbf{v}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ is the sum of the displacements of the first vector on each coordinate axis plus the displacement of the second vector, for each axis -

$$\mathbf{v}_1 + \mathbf{v}_2 = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k}.$$

This is equivalent to moving one vector so that its origin lies at the origin of the coordinate system and then moving the second vector so that its origin is the same point as the terminus of the first vector.



Note that the result is the same if we reverse the order of the addition

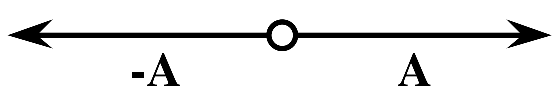
$$\mathbf{v}_1 + \mathbf{v}_2 = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k} = (x_2 + x_1)\mathbf{i} + (y_2 + y_1)\mathbf{j} + (z_2 + z_1)\mathbf{k} = \mathbf{v}_2 + \mathbf{v}_1 .$$

Vector addition is both commutative and associative.

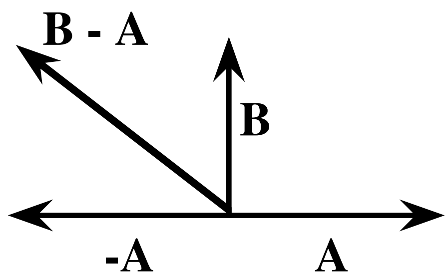
Multiplication of a vector by a scalar: If c is a real number, or a scalar, then the product of the scalar c and the vector $\mathbf{v}$ is a vector in the same direction as $\mathbf{v}$, but with c times the magnitude. This may be expressed as -

$$c\,\mathbf{v} = c(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = cx\,\mathbf{i} + cy\,\mathbf{j} + cz\mathbf{k} = \mathbf{v}c.$$

If $c < 0$, then the direction of the product vector, $c\mathbf{v}$, is opposite to that of $\mathbf{v}$.



Vector Subtraction: If a vector is subtracted then one adds a vector of the same magnitude, but opposite direction. So, $\mathbf{B} - \mathbf{A} = \mathbf{B} + (-\mathbf{A})$.



Distributive Law of Scalar Multiplication: The product of a scalar times a sum of vectors is equal to the sum of the component vectors each multiplied by the same scalar. Basically multiplying a sum of vectors by a scalar is equivalent to scaling all the vectors up, or down, by the same scalar and then adding them. This is probably seen more clearly in terms of symbolic representations:

$$\begin{aligned} c(\mathbf{v_1} + \mathbf{v_2}) &= c((x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k}) \\ &= c(x_1 + x_2)\mathbf{i} + c(y_1 + y_2)\mathbf{j} + c(z_1 + z_2)\mathbf{k} \\ &= (cx_1 + cx_2)\mathbf{i} + (cy_1 + cy_2)\mathbf{j} + (cz_1 + cz_2)\mathbf{k} \\ &= (cx_1\mathbf{i} + cy_1\mathbf{j} + cz_1\mathbf{k}) + (cx_2\mathbf{i} + cy_2\mathbf{j} + cz_2\mathbf{k}) \\ &= c\,\mathbf{v_1} + c\,\mathbf{v_2}\ . \end{aligned}$$

Unit Coordinate Vectors: We have actually been using unit coordinate vectors all along, but we should define them more exactly for future purposes. The coordinate vectors along the positive coordinate axes are generally standardized to be unit vectors in the directions of the coordinate axes. So, $\mathbf{i} = \mathbf{e_x} = \{1,0,0\}$, $\mathbf{j} = \mathbf{e_y} = \{0,1,0\}$, and $\mathbf{k} = \mathbf{e_z} = \{0,0,1\}$. The symbol **e** is often used as it stands for the German word *einheit*, meaning unity, in this case refering to unit vectors. The set of unit coordinate vectors, $\{\mathbf{e_x},\mathbf{e_y},\mathbf{e_z}\}$, is called the basis vectors for the coordinate system that they define. They may be used as an alternative, and somewhat more general way of representing vectors because the $\{\mathbf{e_x},\mathbf{e_y},\mathbf{e_z}\}$ basis vectors are not necessarily mutually perpendicular while the **{i,j,k}** basis vectors generally are -

$$\mathbf{v} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = x\,\mathbf{e_x} + y\,\mathbf{e}_y + z\,\mathbf{e}_z\ .$$

Multiplication of Vectors: The multiplication of vectors is peculiar because there are two different meaning attached to the multiplication of vectors. The reason for this will be clear when we consider quaternions, but for the moment let us define the two meanings of vector multiplication within vector analysis, since they are commonly used.

There are two interpretations of vector multiplication; one yields a scalar, therefore is called the scalar product, and the other yields a vector, therefore is called the vector product. The scalar product is also called the dot product, because of the dot symbol used to indicate the multiplication, and vector product is also called the cross product, again because of the cross symbol used to indicate the multiplication. In quaternion analysis the product of two vectors is the vector product minus the scalar product, a quaternion, but vector analysis can not lead to such a result, because all operations must yield either a scalar or a vector, but not a combination of the two types.

Scalar Product: the dot product or scalar product is defined as follows:

$$\mathbf{v_1} \circ \mathbf{v_2} = x_1 x_2 + y_1 y_2 + z_1 z_2 \,.$$

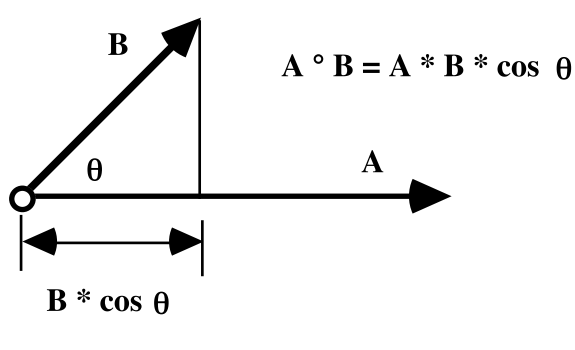
Since the expression does not change if we reverse the order of the multiplication, it follows that the scalar product is commutative and it easy to show that it is also distributive -

$$\begin{aligned}\mathbf{v_1} \circ (\mathbf{v_2} + \mathbf{v}_3) &= x_1(x_2 + x_3) + y_1(y_2 + y_3) + z_1(z_2 + z_3) \\ &= x_1 x_2 + x_1 x_3 + y_1 y_2 + y_1 y_3 + z_1 z_2 + z_1 z_3 \\ &= (x_1 x_2 + y_1 y_2 + z_1 z_2) + (x_1 x_3 + y_1 y_3 + z_1 z_3) \\ &= \mathbf{v_1} \circ \mathbf{v_2} + \mathbf{v_1} \circ \mathbf{v}_3 \,.\end{aligned}$$

From geometry, the formula for the cosine of the angle between two line segments is

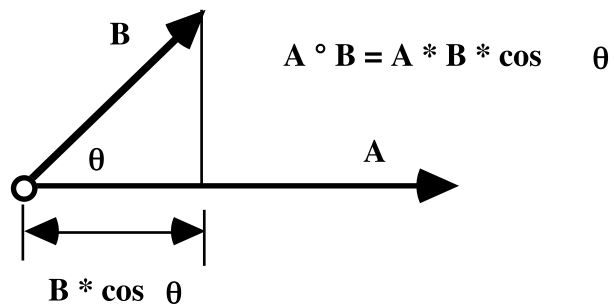
$$\cos\theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} * \sqrt{x_2^2 + y_2^2 + z_2^2}} = \frac{\mathbf{v_1} \circ \mathbf{v_2}}{|\mathbf{v_1}| * |\mathbf{v_2}|}, \text{ therefore}$$

$$\mathbf{v_1} \circ \mathbf{v_2} = |\mathbf{v_1}| * |\mathbf{v_2}| * \cos\theta \,.$$



Note that $|\mathbf{v_2}| * \cos\theta$ is the magnitude of the projection of $\mathbf{v_2}$ upon $\mathbf{v_1}$, therefore the scalar product may be considered to be the magnitude of $\mathbf{v_1}$ times the magnitude of the projection of $\mathbf{v_2}$ upon $\mathbf{v_1}$. . Equivalently, the scalar product may be considered to be the magnitude of $\mathbf{v_2}$

times the magnitude of the projection of $\mathbf{v}_1$ upon $\mathbf{v}_2$. A projection is the distance from the common origin of the two vectors to the point of intersection of a line, drawn through the terminus of the projected vector, that is perpendicular to the second vector.



If $\mathbf{v}_1 \circ \mathbf{v}_2 = 0$ and $\mathbf{v}_1, \mathbf{v}_2 \neq 0$, then $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_2$. Furthermore, $\mathbf{v} \circ \mathbf{v} = |\mathbf{v}|^2$. From these observations we may deduce that -

$$\mathbf{i} \circ \mathbf{i} = \mathbf{j} \circ \mathbf{j} = \mathbf{k} \circ \mathbf{k} = 1, \text{ and}$$
$$\mathbf{i} \circ \mathbf{j} = \mathbf{i} \circ \mathbf{k} = \mathbf{j} \circ \mathbf{k} = 0 .$$

If we write

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}, \text{ then}$$
$$\mathbf{A} = |\mathbf{A}|\left\{\frac{A_x}{|\mathbf{A}|}\mathbf{i} + \frac{A_y}{|\mathbf{A}|}\mathbf{j} + \frac{A_z}{|\mathbf{A}|}\mathbf{k}\right\},$$

which is the product of the magnitude of the vector times a unit vector in the direction of the vector. Now, let

$$cos\,\alpha = \frac{A_x}{|\mathbf{A}|},\ cos\,\beta = \frac{A_y}{|\mathbf{A}|}, \text{ and } cos\ \gamma = \frac{A_z}{|\mathbf{A}|}, \text{ then}$$
$$\mathbf{A} = |\mathbf{A}|\{cos\,\alpha * \mathbf{i} + cos\,\beta * \mathbf{j} + cos\,\gamma * \mathbf{k}\} = |\mathbf{A}| * \mathbf{v} .$$

The vector **v** is a unit vector in the direction of **A** with the components $cos\ \alpha$, $cos\ \beta$, $cos\ \gamma$, the direction cosines of the direction angles α, β, γ. Since $|\mathbf{A}|$ is a scalar, the vector **A** is represented by a scalar times an unit direction vector. For any vector, **B**, the projection of **B** upon **A** is -

$$|\mathbf{B}| * cos\,\theta = \frac{\mathbf{B} \circ \mathbf{A}}{|\mathbf{A}|} = \mathbf{B} \circ \mathbf{v}$$

where θ is the angle between **A** and **B**.

Example: The work, δW, done in moving an object through a displacement, $\boldsymbol{\delta}\mathbf{s}$, is the component of the force, $\mathbf{F}$, in the direction of $\boldsymbol{\delta}\mathbf{s}$ times the displacement -

$$\delta W = |\mathbf{F}| * cos\,\theta * \boldsymbol{\delta}\mathbf{s} = \mathbf{F} \circ \boldsymbol{\delta}\mathbf{s}\,.$$

Vector Product : The vector product is more easily appreciated as a component of quaternion analysis, but it has been usefully abstracted from that context in vector analysis. Given two vectors **A** and **B**, the cross product or vector product of the two vectors is defined to be a vector that is orthogonal to both **A** and **B**, with a magnitude equal to the area of the parallelogram determined by the two vectors. In terms of coordinates -

$$\mathbf{A} \otimes \mathbf{B} = \left(y_A z_B - z_A y_B\right)\mathbf{i} + \left(z_A x_B - x_A z_B\right)\mathbf{j} + \left(x_A y_B - y_A x_B\right)\mathbf{k}$$

which may be expressed also as the determinant

$$\mathbf{A} \otimes \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{vmatrix}$$

There are two possible vectors that are orthogonal to the plane determined by the vectors **A** and **B**. The direction of the cross product is the direction the thumb of the right hand points when its fingers are swept from **A** to **B**. A little experimentation will confirm that

$$\mathbf{B} \otimes \mathbf{A} = -(\mathbf{A} \otimes \mathbf{B}),$$

That is, the vector product is anticommunitive. If one sweeps the fingers of one's right hand from **B** to **A** the thumb points in the opposite direction as it does if the fingers are swept from **A** to **B**. The same result follows from the definition of the cross product in terms of the components, by substituting the **A** terms for the **B** terms and *vice versa*.

Either by substituting in the definition or geometrical construction, one can see that

$$\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}\,.$$

Substitution in the definition or geometrical construction will confirm that

$$n(\mathbf{A} \otimes \mathbf{B}) = (n\,\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (n\mathbf{B})\,.$$

From the definition of the basis vectors it readily follows that

$$\mathbf{i} \otimes \mathbf{i} = \mathbf{j} \otimes \mathbf{j} = \mathbf{k} \otimes \mathbf{k} = 0,$$
$$\mathbf{i} \otimes \mathbf{j} = \mathbf{k} = -(\mathbf{j} \otimes \mathbf{i}),$$
$$\mathbf{j} \otimes \mathbf{k} = \mathbf{i} = -(\mathbf{k} \otimes \mathbf{j}),$$
$$\mathbf{k} \otimes \mathbf{i} = \mathbf{j} = -(\mathbf{i} \otimes \mathbf{k}).$$

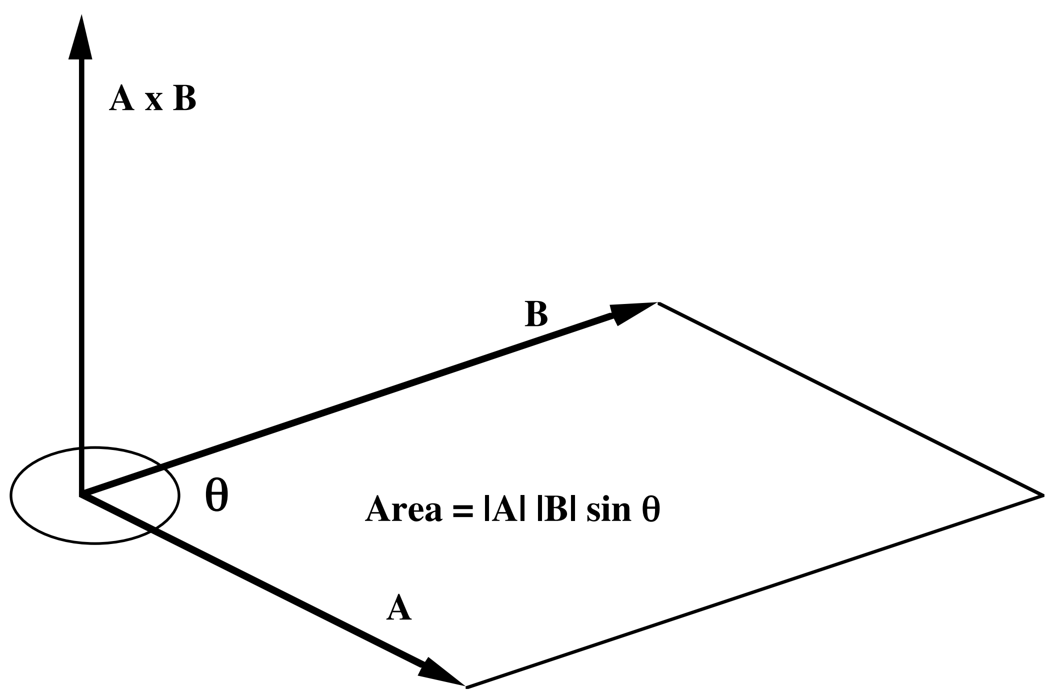
It may be shown that the magnitude of the cross product

$$\begin{aligned} |\mathbf{A} \otimes \mathbf{B}| &= \sqrt{(y_A z_B - z_A y_B)^2 + (z_A x_B - x_A z_B)^2 + (x_A y_B - y_A x_B)^2} \\ &= |\mathbf{A}|^2 * |\mathbf{B}|^2 - (\mathbf{A} \circ \mathbf{B})^2 \\ &= |\mathbf{A}| * |\mathbf{B}| * \sqrt{1 - cos^2\,\theta} \\ &= |\mathbf{A}| * |\mathbf{B}| * sin\,\theta, \end{aligned}$$

where θ is the angle between **A** and **B**. Therefore, we can write the cross product in a more compact and frequently more useful form as

$$|\mathbf{A} \otimes \mathbf{B}| = |\mathbf{A}||\mathbf{B}| sin\,\theta * \boldsymbol{\nu},$$

where $\boldsymbol{\nu}$ is the unit vector normal to the plane of **A** and **B** that completes a right-handed triad. Note that the magnitude of the scalar is the area of the parallelogram that is formed by **A** and **B**.



Example: Let a force, **F**, act at the point **P**(x, y, z) and let the vector **OP** = **r** be the displacement of **P** relative to an origin.

$$\mathbf{OP} = \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

The moment, **N**, or the torque vector about the point **O** is expressed by the cross product

$$\mathbf{N} = \mathbf{r} \otimes \mathbf{F}\,.$$

The magnitude of the moment is $|\mathbf{N}| = |\mathbf{r} \otimes \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}|\,sin\,\theta$ and it is directed along the axis of rotation.

Derivative of a Vector: Consider a vector **A** whose components are a function of a single scalar variable, *u*.

$$\mathbf{A}(u) = A_x(u)\mathbf{i} + A_y(u)\mathbf{j} + A_z(u)\mathbf{k},$$

the derivative of **A** with respect to *u* is

$$\boldsymbol{d}\mathbf{A} \,/\, \boldsymbol{d}\,u = lim_{\Delta\,u->0}\ \Delta\mathbf{A} \,/\, \Delta\,u = lim_{\Delta\,u->0}\left[\frac{\Delta\,A_x(u)}{\Delta\,u}\,\mathbf{i} + \frac{\Delta\,A_y(u)}{\Delta\,u}\,\mathbf{j} + \frac{\Delta\,A_z(u)}{\Delta\,u}\,\mathbf{k}\right],$$

where $\Delta A_x(u) = A_x(u+\Delta u) - A_x(u)$, and so on. Therefore,

$$\frac{\mathbf{dA}}{du} = \frac{\mathbf{dA_x}(u)}{du}\mathbf{i} + \frac{\mathbf{dA_y}(u)}{du}\mathbf{j} + \frac{\mathbf{dA_z}(u)}{du}\mathbf{k}$$

It follows from this equation that

$$\frac{\mathbf{d}(\mathbf{A}+\mathbf{B})}{du} = \frac{\mathbf{d}(\mathbf{A})}{du} + \frac{\mathbf{d}(\mathbf{B})}{du}.$$

Given a position vector, **r**(*t*), that is a function of *t* , it is straight forward to calculate the velocity and acceleration vectors.

Velocity,

$$\mathbf{v}(t) = \frac{\mathbf{dr}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

and *acceleration*

$$\mathbf{a}(t) = \frac{\mathbf{dv}}{dt} = \frac{\mathbf{d^2r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}.$$

Derivatives of Products of Vectors: If n is a scalar, and the vectors **A** and **B** are functions of a single scalar parameter u, then

$$\frac{\boldsymbol{d}(n\mathbf{A})}{du} = lim_{\Delta u \to 0} \frac{n(u+\Delta u)\mathbf{A}(u+\Delta u) - n(u)\mathbf{A}(u)}{\Delta u},$$
$$\frac{d(\mathbf{A} \circ \mathbf{B})}{du} = lim_{\Delta u \to 0} \frac{\mathbf{A}(u+\Delta u) \circ \mathbf{B}(u+\Delta u) - \mathbf{A}(u) \circ \mathbf{B}(u)}{\Delta u},$$
$$\frac{\boldsymbol{d}(\mathbf{A} \otimes \mathbf{B})}{du} = lim_{\Delta u \to 0} \frac{\mathbf{A}(u+\Delta u) \otimes \mathbf{B}(u+\Delta u) - \mathbf{A}(u) \otimes \mathbf{B}(u)}{\Delta u}.$$

The derivative of the first expression may be derived by the following logic.

$$n(u+\Delta u) = n + \Delta n \text{ and } \mathbf{A}(u+\Delta u) = \mathbf{A} + \mathbf{\Delta A}, \text{ therefore}$$
$$\frac{\boldsymbol{d}(n\mathbf{A})}{du} = lim_{\Delta u \to 0} \frac{\{(n+\Delta n)(\mathbf{A} + \mathbf{\Delta A}) - n\mathbf{A}\}}{\Delta u},$$
$$= lim_{\Delta u \to 0} \frac{\{n\mathbf{A} + n*\mathbf{\Delta A} + \Delta n*\mathbf{A} + \Delta n*\mathbf{\Delta A} - n\mathbf{A}\}}{\Delta u},$$
$$= lim_{\Delta u \to 0} \left\{ \frac{n*\mathbf{\Delta A}}{\Delta u} + \frac{\Delta n*\mathbf{A}}{\Delta u} + \frac{\Delta n*\mathbf{\Delta A}}{\Delta u} \right\},$$
$$= \frac{dn(u)}{du} \circ \mathbf{A}(u) + n(u) \circ \frac{\boldsymbol{d}\mathbf{A}(u)}{du}, \text{ thus}$$
$$\frac{\boldsymbol{d}(n\mathbf{A})}{du} = \frac{dn(u)}{du} \circ \mathbf{A}(u) + n(u) \circ \frac{\boldsymbol{d}\mathbf{A}(u)}{du}.$$

By similar manipulation we find the following rules -

$$\frac{d(\mathbf{A} \circ \mathbf{B})}{du} = \frac{\mathbf{dA}(u)}{du} \circ \mathbf{B}(u) + \mathbf{A}(u) \circ \frac{\mathbf{dB}(u)}{du},$$
$$\frac{\mathbf{d(AxB)}}{du} = \frac{\mathbf{dA}(u)}{du} \otimes \mathbf{B}(u) + \mathbf{A}(u) \otimes \frac{\mathbf{dB}(u)}{du}$$

It is essential to maintain the order of the terms in the derivative of the cross product.

The Derivation of Quaternions

At this point we turn to the derivation of quaternions and their use to represent rotations in three dimensional space. However, substantial foundations need to be laid before the concept of quaternions is developed. To start with we will reconsider scalar and vector products, with some differences in notation and the introduction of some new concepts needed for later developments.

Scalar Product Function: The scalar product of one vector, α, into another, β, is denoted by the expression $-S\langle\alpha\beta\rangle$ or $-S\alpha\beta$, a function of two vectors called the *scalar product function or scalar* of α times β. The scalar product of the vectors α and β is constructed by projecting the image of β upon a line parallel with α, that is, by drawing a perpendicular to α that passes through the terminus of β, then multiplying the length of the projection of β times the length of the vector, α. Since the projection of β upon α is $|\beta| * \cos\theta$, where θ is the smaller angle between α and β and $|\beta|$ is the length of β, it follows that $-S\langle\alpha\beta\rangle = |\alpha| * |\beta| * \cos\theta$. This may be rewritten as

$$S\langle\alpha\beta\rangle = |\alpha| * |\beta| * \cos(\pi - \theta).,$$

where the angle between the two vectors is expressed in radians. This definition of the scalar product may seem clumsy, but it is just what is needed later on.

Note that if α and β are perpendicular then

$$\mathrm{S}\langle\alpha\beta\rangle = |\alpha| * |\beta| * \cos\left(\pi - \frac{\pi}{2}\right) = |\alpha| * |\beta| * \cos\left(\frac{\pi}{2}\right) = 0$$

and, if they are parallel, then

$$S\langle\alpha\beta\rangle = |\alpha| * |\beta| * \cos(\pi - 0) = |\alpha| * |\beta| * \cos(\pi) = |\alpha| * |\beta|.$$

Since, $-S\langle\alpha\beta\rangle = |\alpha| * |\beta| * \cos\theta$ is symmetric with respect to α and β it follows that $S\langle\alpha\beta\rangle = S\langle\beta\alpha\rangle$. This can be confirmed geometrically by the use of similar triangles.

Because the sum of the projections of any number of vectors on a line is the projection of their sum, it follows that $S\langle\alpha(\beta + \gamma)\rangle = S\langle\alpha\beta\rangle + S\langle\alpha\gamma\rangle$, therefore the scalar product function is doubly distributive, that is

$$S\left\langle \sum_i \alpha_i * \sum_j \beta_j \right\rangle = \sum_i \sum_j S\left\langle \alpha_i \beta_j \right\rangle.$$

For instance -

$$S\langle(\alpha_1 + \alpha_2)(\beta_1 + \beta_2 + \beta_3)\rangle = S\langle\alpha_1\beta_1\rangle + S\langle\alpha_1\beta_2\rangle + S\langle\alpha_1\beta_3\rangle + S\langle\alpha_2\beta_1\rangle + S\langle\alpha_2\beta_2\rangle + S\langle\alpha_2\beta_3\rangle.$$

Example: Consider the multiplication of two vectors that have been resolved into their components.

$$\alpha = x_\alpha \mathbf{i} + y_\alpha \mathbf{j} + z_\alpha \mathbf{k}, \quad \beta = x_\beta \mathbf{i} + y_\beta \mathbf{j} + z_\beta \mathbf{k},$$

$$\begin{aligned} S\langle\alpha\beta\rangle &= S\langle(x_\alpha \mathbf{i} + y_\alpha \mathbf{j} + z_\alpha \mathbf{k})(x_\beta \mathbf{i} + y_\beta \mathbf{j} + z_\beta \mathbf{k})\rangle \\ &= S\langle x_\alpha \mathbf{i}\, x_\beta \mathbf{i}\rangle + S\langle x_\alpha \mathbf{i}\, y_\beta \mathbf{j}\rangle + S\langle x_\alpha \mathbf{i}\, z_\beta \mathbf{k}\rangle + S\langle y_\alpha \mathbf{j}\, x_\beta \mathbf{i}\rangle + S\langle y_\alpha \mathbf{j}\, y_\beta \mathbf{j}\rangle \\ &\quad + S\langle y_\alpha \mathbf{j}\, z_\beta \mathbf{k}\rangle + S\langle z_\alpha \mathbf{k}\, x_\beta \mathbf{i}\rangle + S\langle z_\alpha \mathbf{k}\, y_\beta \mathbf{j}\rangle + S\langle z_\alpha \mathbf{k}\, z_\beta \mathbf{k}\rangle \\ &= S\langle x_\alpha x_\beta \mathbf{i}^2\rangle + 0 + 0 + 0 + S\langle y_\alpha y_\beta \mathbf{j}^2\rangle + 0 + 0 + 0 + S\langle z_\alpha z_\beta \mathbf{k}^2\rangle \\ &= -(x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta) \end{aligned}$$

An equation such as $S\langle\alpha\beta\rangle = S\langle\gamma\delta\rangle$, implies that the projection of α on β multiplied by the length of β is equal to the projection of γ on δ multiplied by the length of δ, that is $|\alpha| * |\beta| * cos\,\theta = |\gamma| * |\delta| * cos\,\phi$.

Vector Product Function: Assume that the unit of length has been defined, let a vector be drawn at right angles to two given vectors α and β so that rotation about this vector from α to β is positive in a right handed coordinate system, and let the length of the vector be numerically equal to the area of the parallelogram determined by α and β. This vector is denoted by the symbol $\mathbf{V}\langle\alpha\beta\rangle$ or $\mathbf{V}\alpha\beta$, and is called the vector product function or vector of α times β.

Since the area of the parallelogram with α and β as its edges and an angle of θ between the two vectors is $|\alpha| * |\beta| * sin\,\theta$, it follows that the length of the vector $\mathbf{V}\langle\alpha\beta\rangle$ is

$$|\mathbf{V}\langle\alpha\beta\rangle| = |\alpha| * |\beta| * sin\,\theta$$

and the direction of $\mathbf{V}\langle\alpha\beta\rangle$ is perpendicular to the plane that contains α and β pointing in the direction one's thumb points if the curled fingers of one's right hand are swept from α to β. Since it does not change the sign of $\mathbf{V}\langle\alpha\beta\rangle$ to replace θ with $(\pi\text{-}\theta)$, that will be done, to make the two components of the products of vectors agree in their arguments. Therefore, the magnitude of the vector product function

$$|\mathbf{V}\langle\alpha\beta\rangle| = |\alpha| * |\beta| * sin(\pi - \theta).$$

If the vectors are taken in the reverse order, $\mathbf{V}\langle\beta\alpha\rangle$ has the same length as $\mathbf{V}\langle\alpha\beta\rangle$ because the area of the enclosed parallelogram is the same, but the direction is opposite, the rotation being now reversed, so -

$$\mathbf{V}\langle\beta\alpha\rangle = -\mathbf{V}\langle\alpha\beta\rangle.$$

If an equation such as $\mathbf{V}\langle\alpha\beta\rangle = \mathbf{V}\langle\gamma\delta\rangle$ exists, the vectors α, β, γ, and δ must be parallel to the same plane; the areas of the parallelograms determined by α and β and by γ and δ must be equal, and the sense of rotation from α to β must be the same as that from γ to δ.. That is, $|\alpha| * |\beta| * sin(\pi - \theta) = |\gamma| * |\delta| * sin(\pi - \theta)$ and all the vectors lie in the same plane with the sense of the rotation from δ to γ the same as the sense of the rotation from β to α..

Like $S\langle\alpha\beta\rangle$, the function $\mathbf{V}\langle\alpha\beta\rangle$ is a doubly distributive function. This may be demonstrated as follows. If β' is the component of β at right angles to α it is obvious that $\mathbf{V}\langle\alpha\beta\rangle = \mathbf{V}\langle\alpha\beta'\rangle$, and the magnitude of $\mathbf{V}\langle\alpha\beta\rangle$ is equal to the product of the magnitudes of α and β'. This follows from the definition of the vector product function, since $|\beta'| = |\beta| * sin\,\theta$.

$$|\mathbf{V}\langle\alpha\beta\rangle| = |\alpha| * |\beta| * sin(\pi - \theta) = |\alpha| * |\beta'|.$$

If β' and γ' are components of β and γ at right angles to α, and β' and γ' are coplanar in the plane $[\beta', \gamma']$ which is perpendicular to α, then the vectors $\mathbf{V}\langle\alpha\beta\rangle$ and $\mathbf{V}\langle\alpha\gamma\rangle$ will lie in the plane $[\beta', \gamma']$ at right angles to β' and γ' respectively. Since

$$\frac{|\mathbf{V}\langle\alpha\beta'\rangle|}{|\beta'|} = \frac{|\mathbf{V}\langle\alpha\gamma'\rangle|}{|\gamma'|} = |\alpha|,$$

it follows that the triangle formed by β' and γ' is directly similar to the triangle formed by $\mathbf{V}\langle\alpha\beta'\rangle$ and $\mathbf{V}\langle\alpha\gamma'\rangle$. Hence, the vector $\mathbf{V}\langle\alpha\beta'\rangle + \mathbf{V}\langle\alpha\gamma'\rangle$ is at right angles to the vector $\beta' + \gamma'$ and $|\mathbf{V}\langle\alpha\beta'\rangle| + |\mathbf{V}\langle\alpha\gamma'\rangle| = |\beta' + \gamma'| * |\alpha|$ consequently,

$$\mathbf{V}\langle\alpha(\beta' + \gamma')\rangle = \mathbf{V}\langle\alpha\beta'\rangle + \mathbf{V}\langle\alpha\gamma'\rangle.$$

Since we may replace β' with β and γ' with γ without altering the values of the vector products, it follows that :

$$\mathbf{V}\langle\alpha(\beta+\gamma)\rangle = \mathbf{V}\langle\alpha\beta\rangle + \mathbf{V}\langle\alpha\gamma\rangle$$

and

$$\mathbf{V}\langle(\beta+\gamma)\alpha\rangle = \mathbf{V}\langle\beta\alpha\rangle + \mathbf{V}\langle\gamma\alpha\rangle,$$

with α, β, γ being arbitrary vectors.

This relationship may be generalized to

$$\mathbf{V}\left\langle\sum_i \alpha_i * \sum_j \beta_j\right\rangle = \sum_i \sum_j \mathbf{V}\langle\alpha_i\beta_j\rangle,$$

for any number of vectors. In particular, for four vectors,

$$\mathbf{V}\langle(\alpha+\beta)(\gamma+\delta)\rangle = \mathbf{V}\langle(\alpha+\beta)\rangle\gamma + \mathbf{V}\langle(\alpha+\beta)\rangle\delta = \mathbf{V}\langle\alpha\gamma\rangle + \mathbf{V}\langle\beta\gamma\rangle + \mathbf{V}\langle\alpha\delta\rangle + \mathbf{V}\langle\beta\delta\rangle.$$

Note that the order of the vectors must be maintained.

Example: Consider the multiplication of two vectors that have been resolved into their components.

$$\alpha = x_\alpha \mathbf{i} + y_\alpha \mathbf{j} + z_\alpha \mathbf{k}, \quad \beta = x_\beta \mathbf{i} + y_\beta \mathbf{j} + z_\beta \mathbf{k},$$

$$\begin{aligned}
\mathbf{V}\langle\alpha\beta\rangle &= \mathbf{V}\left\langle(x_\alpha \mathbf{i} + y_\alpha \mathbf{j} + z_\alpha \mathbf{k})(x_\beta \mathbf{i} + y_\beta \mathbf{j} + z_\beta \mathbf{k})\right\rangle \\
&= \mathbf{V}\langle x_\alpha \mathbf{i}\, x_\beta \mathbf{i}\rangle + \mathbf{V}\langle x_\alpha \mathbf{i}\, y_\beta \mathbf{j}\rangle + \mathbf{V}\langle x_\alpha \mathbf{i}\, z_\beta \mathbf{k}\rangle + \mathbf{V}\langle y_\alpha \mathbf{j}\, x_\beta \mathbf{i}\rangle + \mathbf{V}\langle y_\alpha \mathbf{j}\, y_\beta \mathbf{j}\rangle \\
&\quad + \mathbf{V}\langle y_\alpha \mathbf{j}\, z_\beta \mathbf{k}\rangle + \mathbf{V}\langle z_\alpha \mathbf{k}\, x_\beta \mathbf{i}\rangle + \mathbf{V}\langle z_\alpha \mathbf{k}\, y_\beta \mathbf{j}\rangle + \mathbf{V}\langle z_\alpha \mathbf{k}\, z_\beta \mathbf{k}\rangle \\
&= \mathbf{V}\langle x_\alpha x_\beta \mathbf{ii}\rangle + \mathbf{V}\langle x_\alpha y_\beta \mathbf{ij}\rangle + \mathbf{V}\langle x_\alpha z_\beta \mathbf{ik}\rangle + \mathbf{V}\langle y_\alpha x_\beta \mathbf{ji}\rangle + \mathbf{V}\langle y_\alpha y_\beta \mathbf{jj}\rangle \\
&\quad + \mathbf{V}\langle y_\alpha z_\beta \mathbf{jk}\rangle + \mathbf{V}\langle z_\alpha x_\beta \mathbf{ki}\rangle + \mathbf{V}\langle z_\alpha y_\beta \mathbf{kj}\rangle + \mathbf{V}\langle z_\alpha z_\beta \mathbf{kk}\rangle \\
&= 0 + \mathbf{V}\langle x_\alpha y_\beta \mathbf{k}\rangle + \mathbf{V}\langle -x_\alpha z_\beta \mathbf{j}\rangle + \mathbf{V}\langle -y_\alpha x_\beta \mathbf{k}\rangle + 0 \\
&\quad + \mathbf{V}\langle y_\alpha z_\beta \mathbf{i}\rangle + \mathbf{V}\langle z_\alpha x_\beta \mathbf{j}\rangle + \mathbf{V}\langle -z_\alpha y_\beta \mathbf{i}\rangle + 0 \\
&= \left[y_\alpha z_\beta - z_\alpha y_\beta\right]\mathbf{i} + \left[z_\alpha x_\beta - x_\alpha z_\beta\right]\mathbf{j} + \left[x_\alpha y_\beta - y_\alpha x_\beta\right]\mathbf{k}
\end{aligned}$$

If $\mathbf{V}\langle\alpha\beta\rangle = 0$ without either α or β equal to zero, then the vector α must be parallel to β, for the area of the parallelogram determined by α and β must vanish. This is because, if $\mathbf{V}\langle\alpha\beta\rangle = |\alpha| * |\beta| * \sin\theta = 0$ and neither $|\alpha|$ or $|\beta|$ is equal to zero, then it follows that $\sin\theta = 0 \Rightarrow \theta = 0, n\pi$, which means that α and β are parallel.

The Multiplication of Vectors: After examining the special product functions associated with vector multiplication, let us now say what shall be meant by vector multiplication. The multiplication of a vector α into a vector β is expressed by the equation,

$$\alpha\beta = S\langle\alpha\beta\rangle + \mathbf{V}\langle\alpha\beta\rangle.$$

Because $\alpha\beta$ is the sum of two doubly distributive components, it is likewise doubly distributive,

$$\sum_i \alpha_i * \sum_j \beta_j = \sum_i \sum_j \alpha_i \beta_j \ .$$

The product $\beta\alpha$ is not generally equal to $\alpha\beta$., since $\beta\alpha = S\langle\alpha\beta\rangle - \mathbf{V}\langle\alpha\beta\rangle$, because $S\langle\beta\alpha\rangle = S\langle\alpha\beta\rangle$ and $\mathbf{V}\langle\beta\alpha\rangle = -\mathbf{V}\langle\alpha\beta\rangle$. Thus, *multiplication of vectors is not commutative*. We speak of $\alpha\beta$ as the product of β by α, or the product of α into β.

Adding and subtracting the expression for the two products $\alpha\beta$ and $\beta\alpha$, we find that

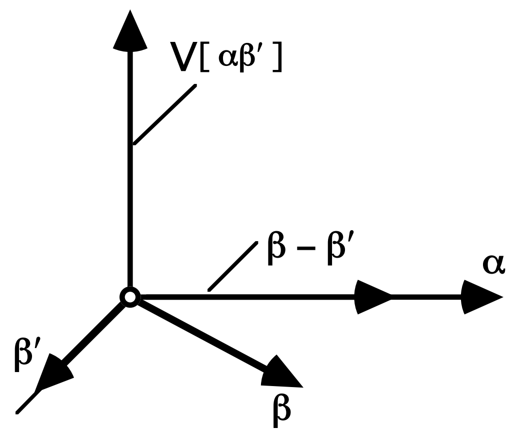
$$S\langle\alpha\beta\rangle = \frac{\alpha\beta + \beta\alpha}{2} \quad \text{and} \quad \mathbf{V}\langle\alpha\beta\rangle = \frac{\alpha\beta - \beta\alpha}{2}.$$

A generalization of this relationship is as follows -

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n$ are vectors and $\boldsymbol{Q} = \prod_1^n \alpha = \alpha_1 \bullet \alpha_2 \bullet \alpha_3 \ldots \bullet \alpha_{n-1} \bullet \alpha_n$ and $\boldsymbol{Q}' = \prod_n^1 \alpha = \alpha_n \bullet \alpha_{n-1} \bullet \alpha_{n-2} \ldots \bullet \alpha_2 \bullet \alpha_1$, then
$\mathrm{S}\langle\boldsymbol{Q}\rangle = \dfrac{\boldsymbol{Q} + (-1)^n \boldsymbol{Q}'}{2}$ and $\mathbf{V}\langle\boldsymbol{Q}\rangle = \dfrac{\boldsymbol{Q} - (-1)^n \boldsymbol{Q}'}{2}$.

Defintion of a Quaternion: The sum of a scalar and a vector is a quaternion, because it is characterized by four independent numbers, a scalar (a) and the three coefficients of the vector component (b, c, d), when resolved along the three basis vectors, $\boldsymbol{q} = \mathrm{a} + \mathrm{b}\mathbf{i} + \mathrm{c}\mathbf{j} + \mathrm{d}\mathbf{k}$.

The product of a pair of vectors is always a quaternion, and conversely, every quaternion may be expressed as the product of two vectors. If $\mathbf{q}$ is a quaternion, if $S\langle q\rangle$ is its scalar part and $\mathbf{V}\langle q\rangle$ is its vector part, so that $\boldsymbol{q} = S\langle q\rangle + \mathbf{V}\langle q\rangle$; if α and β' are two vectors at right angles to one another and to $\mathbf{V}\langle q\rangle$, so that $\mathbf{V}\langle\alpha\beta'\rangle = \mathbf{V}\langle q\rangle$; and if $\beta - \beta'$ is the vector parallel to α, for which $S\langle\alpha(\beta - \beta')\rangle = S\langle q\rangle$, then we have -



$$\begin{aligned}
\boldsymbol{q} &= S\langle \boldsymbol{q}\rangle + \mathbf{V}\langle \boldsymbol{q}\rangle \\
&= S\langle \alpha(\beta-\beta')\rangle + \mathbf{V}\langle \alpha\beta'\rangle \\
&= S\langle \alpha(\beta-\beta') + \alpha\beta'\rangle + \mathbf{V}\langle \alpha(\beta-\beta') + \alpha\beta'\rangle, \text{ since } S\langle\alpha\beta'\rangle = 0 \text{ and } \mathbf{V}\langle \alpha(\beta-\beta')\rangle = 0 \\
&= S\langle \alpha\beta\rangle + \mathbf{V}\langle \alpha\beta\rangle \\
&= \alpha\beta.
\end{aligned}$$

The quaternion has been reduced to the product of a pair of vectors.

This point can also be developed as follows. Define $\boldsymbol{q}$ as -

$$\boldsymbol{q} = a + b\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} = |\alpha|*|\beta|*sin(\pi-\theta) + |\alpha|*|\beta|*cos(\pi-\theta)*\mathbf{n}, \text{ where}$$

$$\mathbf{n} = \frac{b\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}}{g} \quad \text{and} \quad g = \sqrt{b^2+c^2+d^2}.$$

Note that $\mathbf{n}$ is the unit vector in the direction of $\mathbf{V}\langle \boldsymbol{q}\rangle$, If

$$P(\alpha\beta) = |\alpha|*|\beta| \;,$$

then by algebraic and trigonometric manipulation

$$P(\alpha\beta) = \sqrt{a^2+g^2} \text{ and } \theta = arctan(g/a).$$

Therefore, any pair of vectors in the plane perpendicular to $\mathbf{n}$ that satisfy the relations

$$\begin{aligned}
a &= P(\alpha\beta)*sin(\pi-\theta), \\
g &= P(\alpha\beta)*cos(\pi-\theta), \\
\boldsymbol{q} &= a + g\mathbf{n}
\end{aligned}$$

and have the appropriate sense of rotation will yield $\boldsymbol{q}$ as their product. Therefore, the requirements come down to the product of the two vectors lengths being $P(\boldsymbol{\alpha\beta})$ and the angle between them being an angle $(\pi - \theta)$ which is determined by the ratio of the magnitudes of the scalar and vector magnitudes of the quaternion.

Note that in vector notation, $\boldsymbol{q} = -\boldsymbol{\alpha} \circ \boldsymbol{\beta} + \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$, where $\boldsymbol{\alpha} \circ \boldsymbol{\beta}$ is the dot product of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ is the cross-product of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Since quaternions are a combination of scalars and vectors it follows that scalars and vectors may be regarded as simply special cases of quaternions.

The Addition of Quaternions: The sum of any number of quaternions we define to be the sum of their scalar parts plus the sum of their vector parts.

$$\begin{aligned} \boldsymbol{q}_1 + \boldsymbol{q}_2 &= (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\boldsymbol{k}) + (a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\boldsymbol{k}) \\ &= (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\mathbf{j} + (d_1 + d_2)\boldsymbol{k} . \end{aligned}$$

Addition of scalars is associative and commutative, as is the addition of vectors . It follows that the addition of quaternions is also associative and commutative.

Multiplication of a quaternion by a vector: The product of a quaternion and a vector is distributive with respect to the scalar and the vector of the quaternion. Thus

$$\gamma\boldsymbol{q} = \gamma\,(S\langle\boldsymbol{q}\rangle + \mathbf{V}\langle\boldsymbol{q}\rangle) = \gamma\, S\langle\boldsymbol{q}\rangle + \gamma\mathbf{V}\langle\boldsymbol{q}\rangle \text{ and}$$
$$\boldsymbol{q}\gamma = (S\langle\boldsymbol{q}\rangle + \mathbf{V}\langle\boldsymbol{q}\rangle)\gamma = S\langle\boldsymbol{q}\rangle\gamma + \mathbf{V}\langle\boldsymbol{q}\rangle\gamma .$$

The products $\gamma\mathbf{V}\langle\boldsymbol{q}\rangle$ and $\mathbf{V}\langle\boldsymbol{q}\rangle\gamma$ are different quaternions since they are products of vectors. The order of the terms in the product is critical, as always with the multiplications of vectors. The multiplication of a scalar and a vector is commutative, so $\gamma\, S\langle\boldsymbol{q}\rangle = S\langle\boldsymbol{q}\rangle\gamma$.

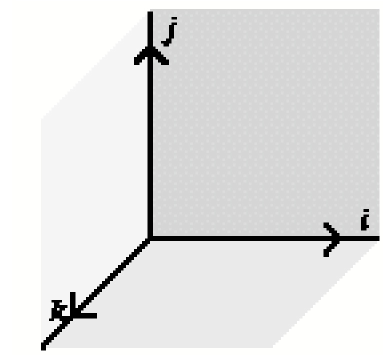
We can interpret expressions such as $\boldsymbol{\alpha}(\boldsymbol{\beta\gamma})$ or $(\boldsymbol{\alpha\beta})\boldsymbol{\gamma}$ (the product of $\boldsymbol{\alpha}$ into the product $\boldsymbol{\beta\gamma}$ and the product $\boldsymbol{\alpha\beta}$ into $\boldsymbol{\gamma}$), and we see that they are distributive with respect to the three vectors, so that

$$\sum_i \alpha_i \left(\sum_j \beta_j * \sum_k \gamma_k \right) = \sum_i \sum_j \sum_k \alpha_i (\beta_j \gamma_k) \text{ and}$$

$$\left(\sum_i \alpha_i * \sum_j \beta_j \right) \sum_k \gamma_k = \sum_i \sum_j \sum_k (\alpha_i \beta_j) \gamma_k$$

We shall now prove that the products are associative, so that we may omit the parentheses, and to this end we shall consider the laws of combination of the three mutually perpendicular unit-vectors, **i**, **j**, and **k**.

Associativity of vector multiplication: Let any three mutually perpendicular unit-vectors **i**, **j**, and **k**, be drawn so that rotation around **i** from **j** to **k** is positive. That is assume an orthogonal right-handed coordinate system with the three basis vectors **i**, **j**, and **k**



According to the usual convention, if **i** and **j** are in the plane of the page, **k** will be directed vertically upwards, and it is seen at once that rotation round **j** from **k** to **i** and round **k** from **i** to **j** is also positive or right-handed.

Because the vectors are mutually perpendicular and of unit length,

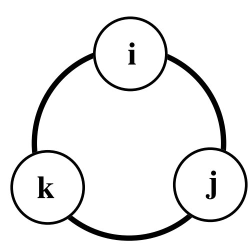
$$S\langle \mathbf{jk}\rangle = S\langle \mathbf{ki}\rangle = S\langle \mathbf{ij}\rangle = 0\ ; \quad S\langle \mathbf{i}^2\rangle = S\langle \mathbf{j}^2\rangle = S\langle \mathbf{k}^2\rangle = -1$$

$$\mathbf{V}\langle \mathbf{jk}\rangle = \mathbf{i}\ ; \quad \mathbf{V}\langle \mathbf{ki}\rangle = \mathbf{j}\ ; \quad \mathbf{V}\langle \mathbf{ij}\rangle = \mathbf{k}\ ; \quad \mathbf{V}\langle \mathbf{kj}\rangle = -\mathbf{i}\ ; \quad \mathbf{V}\langle \mathbf{ik}\rangle = -\mathbf{j}\ ; \quad \mathbf{V}\langle \mathbf{ji}\rangle = -\mathbf{k}\ ;$$

and it follows at once from the definition of vector multiplication that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\ ; \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}\ ; \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}\ ; \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}\, .$$

It is easy to remember these relationships by visualizing the following figure.



If one proceeds clockwise around the circle, then the product of two basis vectors is the third basis vector and if one proceeds counterclockwise, then the product is the negative of the third basis vector.

Example: Consider the multiplication of two vectors that have been resolved into their components.

$$\alpha = x_\alpha \mathbf{i} + y_\alpha \mathbf{j} + z_\alpha \mathbf{k}, \quad \beta = x_\beta \mathbf{i} + y_\beta \mathbf{j} + z_\beta \mathbf{k},$$

$$\begin{aligned} \alpha\beta &= (x_\alpha \mathbf{i} + y_\alpha \mathbf{j} + z_\alpha \mathbf{k})(x_\beta \mathbf{i} + y_\beta \mathbf{j} + z_\beta \mathbf{k}) \\ &= x_\alpha x_\beta \mathbf{ii} + x_\alpha y_\beta \mathbf{ij} + x_\alpha z_\beta \mathbf{ik} + y_\alpha x_\beta \mathbf{ji} + y_\alpha y_\beta \mathbf{jj} + y_\alpha z_\beta \mathbf{jk} + z_\alpha x_\beta \mathbf{ki} + z_\alpha y_\beta \mathbf{kj} + z_\alpha z_\beta \mathbf{kk} \\ &= -x_\alpha x_\beta + x_\alpha y_\beta \mathbf{k} + -x_\alpha z_\beta \mathbf{j} + -y_\alpha x_\beta \mathbf{k} - y_\alpha y_\beta + y_\alpha z_\beta \mathbf{i} + z_\alpha x_\beta \mathbf{j} + -z_\alpha y_\beta \mathbf{i} - z_\alpha z_\beta \\ &= -\left[x_\alpha x_\beta + y_\alpha y_\beta + z_\alpha z_\beta\right] + \left[y_\alpha z_\beta - z_\alpha y_\beta\right]\mathbf{i} + \left[z_\alpha x_\beta - x_\alpha z_\beta\right]\mathbf{j} + \left[x_\alpha y_\beta - y_\alpha x_\beta\right]\mathbf{k} \end{aligned}$$

This is the general case for the quaternion expressed as the product of two vectors. By comparing it with previous examples it is clear that is the sum of the scalar and vector product functions.

Let us now, as in the last section, form the ternary products of these vectors. We have by the relations just given

$$\begin{aligned} &\mathbf{i}(\mathbf{jk}) = \mathbf{i}*\mathbf{i} = -\mathbf{1} = \mathbf{k}*\mathbf{k} = (\mathbf{ij})\mathbf{k} = \mathbf{ijk}, \\ &(\mathbf{i}^2)\mathbf{j} = -\mathbf{j} = \mathbf{i}*\mathbf{k} = \mathbf{i}(\mathbf{ij}) = \mathbf{i}^2\mathbf{j}, \\ &\mathbf{i}(\mathbf{j}^2) = -\mathbf{i} = \mathbf{k}*\mathbf{j} = (\mathbf{ij})\mathbf{j} = \mathbf{ij}^2, \end{aligned}$$

the parentheses being omitted as they are seen to be unnecessary. Similarly, for every ternary product of **i**, **j**, and **k**, the parentheses may be shown to be unnecessary.

For quaternary products, for example, let $\boldsymbol{\iota}, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\mu}$ each denote some one of the three symbols **i**, **j**, **k**, then $\boldsymbol{\iota}(\boldsymbol{\kappa\lambda\mu}) = \boldsymbol{\iota}(\boldsymbol{\kappa}(\boldsymbol{\lambda\mu})) = \boldsymbol{\iota}\ \boldsymbol{\kappa}(\boldsymbol{\lambda\mu}) = ((\iota\kappa)\lambda)\mu = (\iota\kappa\lambda)\mu = \boldsymbol{\iota\kappa\lambda\mu}$, because, for example, $\boldsymbol{\iota}(\boldsymbol{\kappa}(\boldsymbol{\lambda\mu}))$ is a ternary product, as $\boldsymbol{\lambda\mu}$ must be ±**i**, ±**j**, ±**k**, of -1. In this way all products of the symbols **i**, **j**, **k** are seen to be associative.

It is a useful exercise to show that the associative law enables us to deduce all the relations from Hamilton's fundamental formula,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}.$$

For example, **i . ijk** = -**i** gives **jk** = **i**.

We can now show that multiplication of vectors is associative. Let any three vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ be expressed in terms of **i**, **j**, **k**, so that

$$\boldsymbol{\alpha} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad \boldsymbol{\beta} = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}, \quad \boldsymbol{\gamma} = x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k}.$$

then,

$$\boldsymbol{\alpha}(\boldsymbol{\beta\gamma}) = \sum\sum\sum x\,\mathbf{i}(y'\,\mathbf{j} * z''\mathbf{k}) = \sum\sum\sum xy'z''\mathbf{i}(\mathbf{jk}) = \sum\sum\sum xy'z''\mathbf{ijk},$$
$$(\boldsymbol{\alpha\beta})\boldsymbol{\gamma} = \sum\sum\sum (x\mathbf{i} * y'\,\mathbf{j})z''\,\mathbf{k} = \sum\sum\sum xy'z''(\mathbf{ij})\mathbf{k} = \sum\sum\sum xy'z''\mathbf{ijk},$$

so that $\boldsymbol{\alpha}\ (\boldsymbol{\beta\gamma}) = (\boldsymbol{\alpha\beta})\ \boldsymbol{\gamma} = \boldsymbol{\alpha\beta\gamma}$, and similarly for all products of higher orders.

Therefore, multiplication of quaternions is associative, because a quaternion may always be expressed as the product of a pair of vectors. The product of any number of vectors, taken in any given order, is a quaternion.

The Division of Vectors: The division of vectors may be reduced to multiplication by the inverse or reciprocal of the quaternion in the divisor. The definition of a product of vectors says that the square of a vector is

$$\mathbf{a}^2 = S\langle\mathbf{a}^2\rangle = -|\mathbf{a}|^2,$$

since the vector is necessarily parallel with itself,therefore

$$\mathbf{V}\langle\mathbf{a}^2\rangle = 0$$

and

$$S\langle\mathbf{a}^2\rangle = -|\mathbf{a}| * |\mathbf{a}| * cos(0) = -|\mathbf{a}|^2.$$

This is remarkable since it says that, whereas a product of vectors is generally a quaternion with both scalar and vector parts, the square of a vector is always a scalar.

It follows from the above relation that

$$-\frac{\mathbf{a}}{|\mathbf{a}|^2} * \mathbf{a} = 1,$$

and thus

$$\frac{1}{\mathbf{a}} = \mathbf{a}^{-1} = \frac{-\mathbf{a}}{|\mathbf{a}|^2}$$

is the reciprocal of a vector **a**. The vector $\mathbf{a}^{-1}$ is opposite in direction and its length is the reciprocal of **a**. We can therefore interpret products such as $\beta\alpha^{-1}$ and $\alpha^{-1}\beta$, and the first of these we shall call the quotient of β by α, and denote it by β/α or $\beta{:}\alpha$.

The reciprocal of any product of vectors is the product of their reciprocals taken in the reverse order. For instance, if $Q = \alpha\beta\gamma\delta$ and $Q' = \delta^{-1}\gamma^{-1}\beta^{-1}\alpha^{-1}$, then $QQ' = 1 = Q'Q$.

$$QQ' = \alpha\beta\gamma\delta\delta^{-1}\gamma^{-1}\beta^{-1}\alpha^{-1} = \alpha\beta\gamma\gamma^{-1}\beta^{-1}\alpha^{-1} = \alpha\beta\beta^{-1}\alpha^{-1} = \alpha\alpha^{-1} = 1$$
$$Q'Q = \delta^{-1}\gamma^{-1}\beta^{-1}\alpha^{-1}\alpha\beta\gamma\delta = \delta^{-1}\gamma^{-1}\beta^{-1}\beta\gamma\delta = \delta^{-1}\gamma^{-1}\gamma\delta = \delta^{-1}\delta = 1$$

Similarly, *the reciprocal of a product of quaternions is the product of the quaternions taken in the reverse order*. The argument is as follows:

$$p = \alpha\beta, \quad q = \gamma\delta, \text{ thus } pq = \alpha\beta\gamma\delta$$
$$(pq)^{-1} = \delta^{-1}\gamma^{-1}\,\beta^{-1}\alpha^{-1} = \gamma\delta\,\alpha\beta = qp.$$

Hence, every quotient of vectors or of quaternions is a quaternion; and more generally, every combination of quaternions by the processes of addition, subtraction, multiplication and division is a quaternion.

Note that $(\delta/\gamma)(\beta/\alpha) \neq (\delta\beta)/(\gamma\alpha)$ since this may be rewritten as $\delta\gamma^{-1}\beta\alpha^{-1} \neq \delta\beta\gamma^{-1}\alpha^{-1}$.. Also

$$(\beta\gamma)/(\alpha\gamma) = (\beta\gamma)(\alpha\gamma)^{-1} = \beta\gamma\gamma^{-1}\alpha^{-1} = \beta/\alpha,$$

whereas

$$(\gamma\beta)/(\gamma\alpha) = (\gamma\beta)(\gamma\alpha)^{-1} = \gamma\beta\alpha^{-1}\gamma^{-1} = \gamma(\beta/\alpha)\gamma^{-1} \ .$$

Therefore, one must be careful in interpreting expressions, since the order of the terms is critical.

The Conjugate of a Quaternion: The conjugate $K\langle q\rangle$ of a quaternion q is defined by the relation $K\langle q\rangle = S\langle q\rangle - \mathbf{V}\langle q\rangle$. If then $q = \alpha\beta$, we have $K\langle q\rangle = \beta\alpha$, and

$$q\mathbf{K}\langle q\rangle = \alpha\beta\beta\alpha = |\beta^2| * \alpha^2 = |\alpha^2| * |\beta^2| = |\alpha^2| * \beta^2 = \beta\alpha\alpha\beta = K\langle q\rangle q$$

This is remarkable in the sense that the product of these particular vectors is a scalar whereas products of vectors are usually quaternions. As you can see the reason that the product of a quaternion and its conjugate is a scalar is that it can be resolved into the product of two vector squares.

The product of the lengths of the vectors into which a quaternion is resolvable is therefore independent of any particular selection of the vectors because $S\langle q\rangle$ and $\mathbf{V}\langle q\rangle$ are independent of any particular pair of vectors; and the square of this product is

$$qK\langle q\rangle = K\langle q\rangle q = (S\langle q\rangle + \mathbf{V}\langle q\rangle)(S\langle q\rangle - \mathbf{V}\langle q\rangle) = S\langle q\rangle^2 - \mathbf{V}\langle q\rangle^2 = |q|^2 = |\alpha|^2 * |\beta|^2,$$

and we call the constant product of amplitudes, $|q|$, the *amplitude* or *length* of the quaternion. If the quaternion is expressed in terms of its components

$$q = a + b\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k} \;,$$

then ,

$$|q|^2 = S\langle q\rangle^2 + \mathbf{V}\langle q\rangle^2 = a^2 + \left(b^2 + c^2 + d^2\right),$$
$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2}\,.$$

The conjugate is important for division of quaternions, because we can evaluate a quotient by multiplying the numerator and denominator by the conjugate of the denominator, yielding a scalar equal to the inverse of the square of the amplitude of the denominator times the product of the numerator and the conjugate of the denominator. Thus a division may be converted to multiplication, which yields a product that is a quaternion times a scalar.

$$\frac{a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}}{a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}} = \frac{a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}}{a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}} * \frac{a_2 - b_2\mathbf{i} - c_2\mathbf{j} - d_2\mathbf{k}}{a_2 - b_2\mathbf{i} - c_2\mathbf{j} - d_2\mathbf{k}}$$
$$= \frac{\left(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}\right) * \left(a_2 - b_2\mathbf{i} - c_2\mathbf{j} - d_2\mathbf{k}\right)}{a_2^2 + b_2^2 + c_2^2 + d_2^2}$$

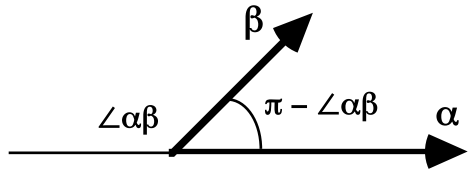
As with vectors, one can express any quaternion as the product of an scalar amplitude and a unit quaternion,

$$q = \alpha\beta = |\alpha| * \frac{\alpha}{|\alpha|} * |\beta| * \frac{\beta}{|\beta|} = |\alpha| * |\beta| * \left[\frac{\alpha}{|\alpha|}\frac{\beta}{|\beta|}\right] = |q| * U\langle q\rangle,$$
$$U\langle q\rangle = \frac{\alpha}{|\alpha|}\frac{\beta}{|\beta|} = \frac{\alpha\beta}{|\alpha| * |\beta|}\,.$$

The unit quaternion, $\boldsymbol{U}\langle \boldsymbol{q} \rangle$, is also called the *versor* of the quaternion and the amplitude of the quaternion, $|\boldsymbol{q}|$, is also called the *tensor* of the quaternion.

If $(\pi - \angle \boldsymbol{q}) = (\pi - \angle \boldsymbol{\alpha\beta})$ is the angle between the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which is less that two right angles and measured from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$, we see by the definitions of $S\boldsymbol{q}$ and $\mathbf{V}\boldsymbol{q}$ that

$$S\langle \boldsymbol{q} \rangle = |\boldsymbol{q}| * \cos \angle \boldsymbol{q} \quad \text{and} \quad |\mathbf{V}\langle \boldsymbol{q} \rangle| = |\boldsymbol{q}| * \sin \angle \boldsymbol{q} .$$



The angle $\angle \boldsymbol{q}$ is called the angle of the quaternion, and is independent of any particular set of vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

This definition of the angle of the quaternion follows from the analyses that were given above, thus the angle of the quaternion is 180° or π radians minus the angular excursion between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ that is less than 180° or π radians. It is directed from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$. This definition is a bit cumbersome, but it will become fairly natural when the quaternion is reinterpreted in the next section.

A plane at right angles to $\mathbf{V}\langle \boldsymbol{q} \rangle$ is called the *plane* of the quaternion and $\boldsymbol{U}\langle \mathbf{V}\langle \boldsymbol{q} \rangle \rangle$ is called the *axis*.

Let $\boldsymbol{q} = w + x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then -

$$\begin{aligned}
\boldsymbol{K}\langle \boldsymbol{q} \rangle &= w - x\,\mathbf{i} - y\,\mathbf{j} - z\,\mathbf{k}, \\
|\boldsymbol{q}| &= \sqrt{w^2 + x^2 + y^2 + z^2} = h, \\
|\mathbf{V}\langle \boldsymbol{q} \rangle| &= \sqrt{x^2 + y^2 + z^2} = g, \\
\boldsymbol{U}\langle \mathbf{V}\langle \boldsymbol{q} \rangle \rangle &= \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{g}, \\
\boldsymbol{U}\langle \boldsymbol{q} \rangle &= \frac{w + x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{h}, \\
|\mathbf{V}\langle \boldsymbol{U}\langle \boldsymbol{q} \rangle \rangle| &= \sqrt{g/h} .
\end{aligned}$$

Quaternions as the Ratio of Vectors: A quaternion can always be expressed as a quotient or ratio of vectors, because we can always reduce it to the product of two vectors and one of the vectors may be the inverse of another vector. Therefore, if one writes the expression for the ratio of β to α (see figure, below) -

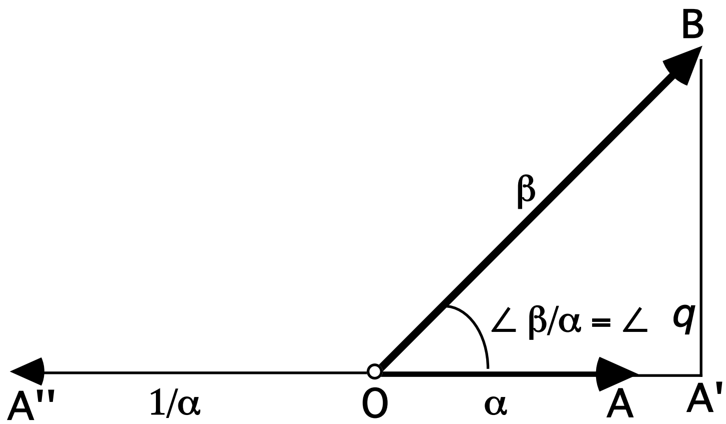
$$q = \frac{\beta}{\alpha} = \frac{\mathbf{OB}}{\mathbf{OA}}$$

it follows that -

$$|q| = \frac{|OB|}{|OA|} \quad \text{and} \quad U\langle q\rangle = \frac{\mathbf{U}\langle\mathbf{OB}\rangle}{\mathbf{U}\langle\mathbf{OA}\rangle}, \text{ therefore}$$

$$S\langle q\rangle = \frac{\mathbf{OA}'}{\mathbf{OA}} \quad \text{and} \quad \mathbf{V}\langle q\rangle = \frac{\mathbf{A'B}}{\mathbf{OA}} \quad \text{and} \quad \angle q = \mathbf{AOB},$$

the line BA' being drawn perpendicular to OA.



These relations may be deduced as follows. $q = \beta/\alpha = \mathbf{OB}/\mathbf{OA}$. Since a quaternion can always be expressed as a product of two vectors, it can be expressed as the product $q = \beta\alpha^{-1}$, which is $\mathbf{OB} \bullet \mathbf{OA}''$. The vector $\alpha^{-1} = \mathbf{OA}''$ has a length $|\mathbf{OA}''| = 1/|\mathbf{OA}|$ and a direction that is opposite to that of $\mathbf{OA}$. Consequently,

$$|q| = |\beta| * |\alpha^{-1}| = \frac{|\mathbf{OB}|}{|\mathbf{OA}|}$$

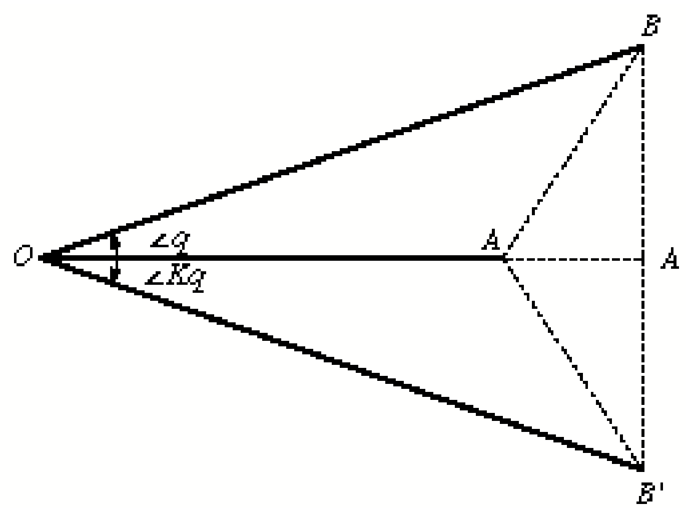
and

$$U\langle q\rangle = \mathbf{U}\langle\beta\rangle \bullet \mathbf{U}\langle\alpha^{-1}\rangle = \frac{\mathbf{U}\langle\mathbf{OB}\rangle}{\mathbf{U}\langle\mathbf{OA}\rangle}.$$

The scalar part of q the projection of β upon the line along the axis of α and α^{-1} (i.e. $\mathbf{OA}'$) times the length of $1/\alpha$ (i.e. $1/\mathbf{OA}$), which leads to $S\langle q\rangle = \mathbf{OA}'/\mathbf{OA}$. The vector part of q is the projection of β on a perpendicular to the line through α and α^{-1}, or $A'B$, times the length of $1/\alpha$, therefore $\mathbf{V}\langle q\rangle = A'B/OA$. Finally, the angle of the quaternion q, $\angle q$, is the angle remaining when the angle between β and $1/\alpha$ is subtracted from π, which is the angle between α and β, therefore $\angle q = \mathbf{AOB}$. Now the angle of the quaternion is the angle between the two vectors that form the quotient. This is one reason that defining the quaternion in this manner is more convenient. Notice that the angle of the quaternion when the quaternion is expressed as a ratio of vectors is the angle between the two vectors, measured from the vector in the denominator to the vector in the numerator.

The shape of the triangle **AOB** is constant for a given quaternion. For this reason quaternions were called ratio of vectors by Hamilton, as they depend upon the relative magnitudes and relative directions of the vectors that are multiplied to obtain them.

It is not difficult to show that the conjugate $K\langle q\rangle$ is the mirror reflection about the vector in the denominator of the ratio, because $q + K\langle q\rangle = 2*S\langle q\rangle$ and $q - K\langle q\rangle = 2*\mathbf{V}\langle q\rangle$. In Hamilton's nomenclature the triangle formed by the component vectors of $K\langle q\rangle$ is said to be inversely similar to the triangle formed by the component vectors of q. That is they are mirror images of each other.



It can be shown that

$$pKq + qKp = 2*S\langle pKq\rangle = 2*S\langle qKp\rangle$$

and

If $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \ldots, \boldsymbol{p}_n$ are quaternions and $\boldsymbol{Kp}$ is the conjugate of $\boldsymbol{p}$, then

$$\mathrm{S}\langle \boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\ldots\boldsymbol{p}_n\rangle = \frac{\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\ldots\boldsymbol{p}_n + \boldsymbol{Kp}_n\boldsymbol{Kp}_{n-1}\boldsymbol{Kp}_{n-2}\ldots\boldsymbol{Kp}_1}{2} \quad \text{and}$$

$$\mathbf{V}\langle \boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\ldots\boldsymbol{p}_n\rangle = \frac{\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\ldots\boldsymbol{p}_n - \boldsymbol{Kp}_n\boldsymbol{Kp}_{n-1}\boldsymbol{Kp}_{n-2}\ldots\boldsymbol{Kp}_1}{2}$$

Quaternions as Rotational Operators: Notice that the value of a quaternion depends upon the lengths and the angle between the two vectors that are multiplied or divided to produce it. It does not depend on the location of the origins or termini of the vectors, only on their magnitudes and relative directions. Therefore there are an infinity of vector pairs that might produce a given quaternion if multiplied together. However, once one of the vectors is determined the other is as well, since it must stand in a particular relation to the first and have a particular magnitude.

$$\frac{\beta}{\alpha} = \boldsymbol{q} \quad \Leftrightarrow \quad \beta = \boldsymbol{q}\alpha \quad \Leftrightarrow \quad \alpha = \boldsymbol{q}^{-1}\beta$$

If $\boldsymbol{q}\alpha = \beta$ then it follows that α and β must both lie in the plane of the quaternion, that is the plane perpendicular to the vector component of the quaternion $\boldsymbol{q}$, thus both will be perpedicular to $\mathbf{V}\langle \boldsymbol{q}\rangle$. In addition, α and β will be separated by the angle of the quaternion, $\angle\boldsymbol{q}$. Since

$$|\boldsymbol{q}| = \frac{|\beta|}{|\alpha|} \quad \Rightarrow \quad |\beta| = |\alpha| * |\boldsymbol{q}| \;\text{ and }\; |\alpha| = \frac{|\beta|}{|\boldsymbol{q}|}$$

the magnitude of the unspecified vector will be determined uniquely. If we look at the quaternion as an operator, then it takes a vector, α, in its plane and turns it through an angle of $\angle\boldsymbol{q}$ and multiplies its length by the magnitude of the quaternion. It follows that there is a quaternion that will rotate and vector, α, into any other vector, β, namely $\boldsymbol{q} = \beta/\alpha = \beta\alpha^{-1}$.

One place that we frequently use this property of quaternions is using the unit quaternion, $\boldsymbol{U}\langle \boldsymbol{q}\rangle$, to rotate vectors about an axis, the direction or versor of the quaternion, without changing their magnitude. Since the magnitude of the quaternion, $|\boldsymbol{q}|$, changes the magnitude of the transformed vector by a constant ratio, we can visualize the operator quaternion as a two step process -

$$\beta = \boldsymbol{q}\alpha \quad \Leftrightarrow \quad \beta = \boldsymbol{U}\langle \boldsymbol{q}\rangle \bullet |\boldsymbol{q}| * \alpha = |\boldsymbol{q}| * \boldsymbol{U}\langle \boldsymbol{q}\rangle \bullet \alpha$$

Multiplication of Quaternions: Up to this point we have been concerned with the multiplication of vectors and the the quaternions that they produce. Now, consider the consequnces of multiplying quaternions. We have already established that the product of quaterion multiplication will always be a quaternion and similarly for division since we can always find the inverse of a quaternion and multiply by the inverse.

Assume that two quaternions, ***q*** and ***r***, are multiplied, ***s* = *rq***. Each quaternion may be represented as the product of two vectors and one can move the planes of the quaternions so that they intersect along the line **OB**, which contains the vector common to both sets vector products. In the plane of ***r*** we construct the other quotient of the vector product, **OC**, and similarly we construct the other quotient of ***q***, **OA**, in its plane. Now ***r* = OC/OB** and ***q* = OB/OA**, therefore -

$$\boldsymbol{s} = \boldsymbol{rq} = \frac{\boldsymbol{OC}}{\boldsymbol{OB}}\frac{\boldsymbol{OB}}{\boldsymbol{OA}} = \boldsymbol{OC} \bullet \boldsymbol{OB}^{-1} \bullet \boldsymbol{OB} \bullet \boldsymbol{OA}^{-1} = \boldsymbol{OC} \bullet \boldsymbol{OA}^{-1} = \frac{\boldsymbol{OC}}{\boldsymbol{OA}}.$$

The two vectors **OA** and **OC** determine a new plane, which is the plane of the quaternion, ***s***, the angle of the quaternion, $\angle\boldsymbol{s}$, is the angle between **OC** and **OA**, and the magnitude of ***s*** is the ratio of the magnitude of the vector **OC** to the magnitude of the vector **OA** .

This elegant geometrical construction gives some idea of the relationships that exist between quaternions and their product and why they interact as they do. If we use the rotation operator interpretation, then one can see that the result of transforming a vector with one quaternion and then a second, that is rotating it through the angle $\angle\boldsymbol{q}$ about the vector axis $\mathbf{V}\langle \boldsymbol{q}\rangle$ and then through the angle $\angle\boldsymbol{r}$ about the vector axis $\mathbf{V}\langle \boldsymbol{r}\rangle$, is the same as rotating it through the angle $\angle\boldsymbol{s}$ about the vector axis $\mathbf{V}\langle \boldsymbol{s}\rangle$. The magnitude of the transformed vector is $|\boldsymbol{s}| = \frac{|\boldsymbol{OC}|}{|\boldsymbol{OA}|}$.

The product ***t* = *qr*** can be obtained by a similar construction. In this instance, we construct the conjugate vectors for **OA** and **OC**, that is, the vectors that are mirror images across **OB** of **OA** and **OC**, each in its respective plane. If **OA′** is the conjugate of **OA** and **OC′** is the conjugate of **OC**, then -

$$\boldsymbol{t} = \boldsymbol{qr} = \frac{\boldsymbol{OA}'}{\boldsymbol{OB}}\frac{\boldsymbol{OB}}{\boldsymbol{OC}'} = \boldsymbol{OA}' \bullet \boldsymbol{OB}^{-1} \bullet \boldsymbol{OB} \bullet \boldsymbol{OC}' = \boldsymbol{OA}' \bullet \boldsymbol{OC}' = \frac{\boldsymbol{OA}'}{\boldsymbol{OC}'}$$

In general, it can be shown

$$\mathrm{S}\langle \boldsymbol{pq}\rangle = \mathrm{S}\langle \boldsymbol{qp}\rangle\,; \quad |\mathbf{V}\langle \boldsymbol{pq}\rangle| = |\mathbf{V}\langle \boldsymbol{qp}\rangle|\,; \quad \text{and } \angle\boldsymbol{pq} = \angle\boldsymbol{qp}\,, \text{ but } \mathbf{V}\langle \boldsymbol{pq}\rangle \neq \mathbf{V}\langle \boldsymbol{qp}\rangle.$$

It can be shown that

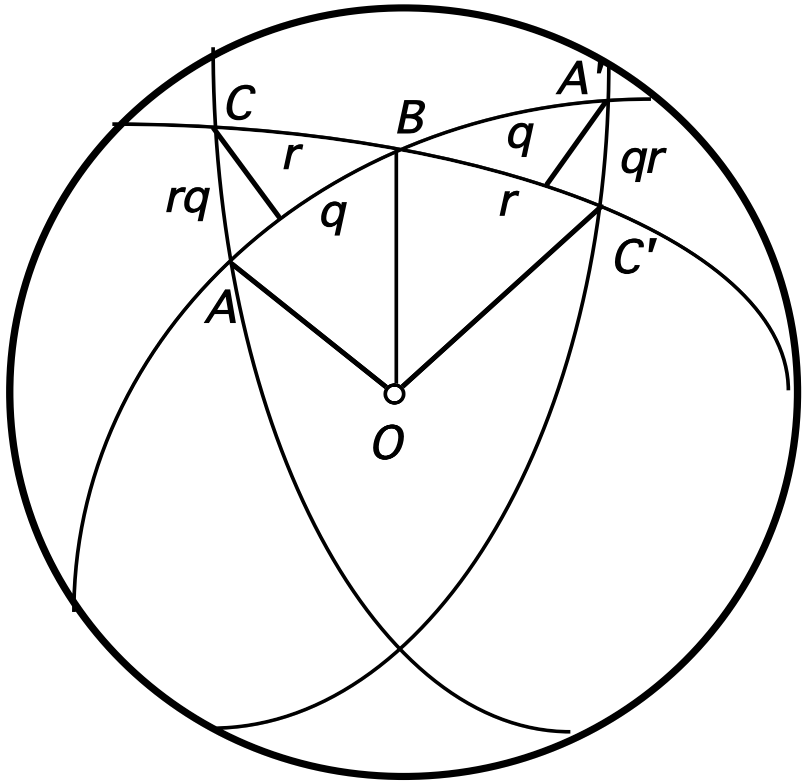
$$|\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{p}_3\ldots\boldsymbol{p}_n| \text{ is independent of the order of the terms },$$

meaning that the overall magnification that occurs with a series of rotations does not depend upon the order in which the rotations are carried out. It can also be shown that

$$\boldsymbol{U}\langle \boldsymbol{p}_1 \boldsymbol{p}_2 \boldsymbol{p}_3 \ldots \boldsymbol{p}_n \rangle = \boldsymbol{U}\langle \boldsymbol{p}_1 \rangle \boldsymbol{U}\langle \boldsymbol{p}_2 \rangle \boldsymbol{U}\langle \boldsymbol{p}_3 \rangle \ldots \boldsymbol{U}\langle \boldsymbol{p}_n \rangle ,$$

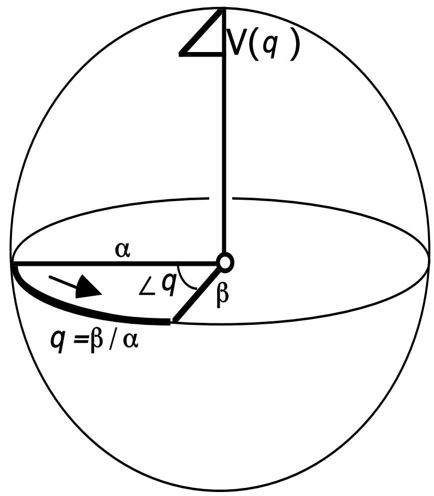
which means that the overall direction after a series of rotations is the same as if one multiplied the unit quaternions of the rotations, in the same order.

Interpretation of Unit Quaternions as Great Circle Arcs: For many purposes one wants to use the quaternions as rotational operators, to express movements in three dimensional space. For those purposes, it is possible to set the magnitude of the quaternion equal to unity and deal only with unit quaternions. The multiplication of unit quaternions may be interpreted very effectively as movements along great arcs on a unit sphere. The intersection of the plane of the quaternion with an unit sphere centered upon the origin of the vector component of the unit quaternion is a great circle on that sphere. An arc along that great circle may be expressed as an angular excursion. For instance, the arc ***BC*** on the subjacent figure is codified as the quaternion ***r***, for which the magnitude is 1.0 and the angle of the quaternion is $\angle \boldsymbol{r}$ which will be interpreted as extending from B to C. The quaternion r could have been equally validly interpreted as extending from C′ to B, or any other equally long arc on that great circle. If we construct another quaternion, ***q***, in the same manner, extending from A to B or B to A′, then it is possible to represent the product ***rq*** by an arc of another great circle, extending from A to C.



The quaternion extends from A to C because one first traverses from A to B (***q***) and then from B to C (***r***) because those are the directions of those quaternion arcs. When interpreting a product one starts with the rightmost operator and then evokes successive operators progressing to the left. Remember that the order of the operators is critical when dealing with quaternions or rotations. If we traverse the arc from C′ to B (***r***) and then the arc from B to A′ (***q***) then the result is the great circle arc from C′ to A′. As has already been established, but can be easily visualized here, the quaternion ***rq*** is not the same as the quaternion ***qr***.

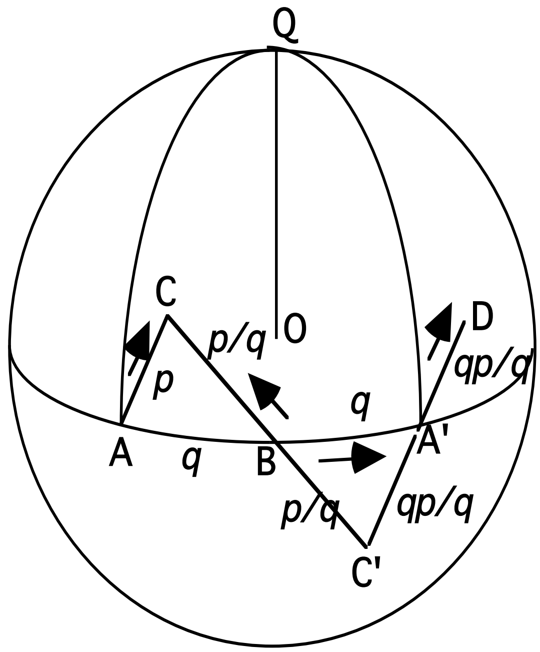
If we abstract an unit quaternion from the unit sphere then it might appear similar to the following figure.



The quaternion, $\boldsymbol{q} = \beta/\alpha$, is an arc of the unit circle bounded by the termini of the two vectors α and β, which define the plane of the quaternion, The vector component of the quaternion is of unit length and it extends perpendicular to the plane of the quaternion, at the common origin of the two vectors. The angle of the quaternion lies between α and β , extending from α to β.

Consider a special case, that that exists when ***r*** and ***q*** are coplanar, that is , when they are arcs on the same great circle. That is the one situation in which $\boldsymbol{rq} = \boldsymbol{qr}$ for all appropriate ***r*** and ***q***. One can see this readily, because as long as the arcs are along the same great circle it does not matter if one traverses the arc specified by ***r*** and then the arc specified by ***q*** or vice versa. In either case one has traversed the same length of the great circle's arc. Remember that the quaternion is not specific as to which particular segment of the great circle is included, only of how much of the great circle is involved.

Rotations of Arbitrary Vectors about a Specific Quaternion Axis: When considering the rotation of vectors in the plane of a quaternion it was observed that $\beta = \boldsymbol{q}\alpha$ transforms the vector α by rotating it through the angle of the quaternion $\angle\boldsymbol{q}$. In the following, we take up the problem of the transformation of an arbitrary vector ρ by rotation about the axis of the vector component ($\mathbf{V}\langle \boldsymbol{q} \rangle$)of the quaternion ***q***. In a way, this represents the culmination of our effort to describe the formal properties of quaternions, because this is the attribute of quaterions that will be most commonly used in describing movements of orientable objects.



The above figure resembles the earlier figure except that we start with the arc ***AC*** which corresponds to the quaternion ***p***. By construction, we produce the similar triangles ABC and A′BC′ and setting the arc ***A′D*** equal in length to ***C′A′*** along the same great circle we establish that ***A′D*** is the result of rotating ***AC*** through an arc of ***AB*** twice. If we call the arc ***AB*** the quaternion ***q***, then ***BA′*** is also ***q***. If ***AC = p*** and ***AB = q***, then, by division, ***BC = p/q***, as does ***C′B***, by similar triangles. For triangle ABC we can check this by observing that $AC = (BC)(AB) = (p/q)q = pq^{-1}q = p$. If we write down the expression for triangle A′BC′ we find that it yields $C'A' = (BA')(C'B) = q(p/q) = qpq^{-1}$. Since $A'D = C'A'$, by the original construction, it follows that $A'D = qpq^{-1}$. Therefore, for an arbitrary vector ρ, the vector component of a quaternion ***p*** $\left(\rho = \mathbf{V}\langle p \rangle\right)$, its transform after being rotated through an angle of $2{*}\angle q$ is given by $\rho \Rightarrow q\rho q^{-1}$.

Example: Let **OP** be a vector from a fixed point to a point on a rigid body. The rigid body is rotated about the axis $\mathbf{OQ} = \mathbf{V}\langle q \rangle$ through an angle of $2{*}\angle q$. The point P is carried to the point P′, where

$$\mathbf{OP}' = q \bullet \mathbf{OP} \bullet q^{-1}.$$

The displacement produced by the rotation is

$$\mathbf{PP}' = \left(q \bullet \mathbf{OP} \bullet q^{-1}\right) - \mathbf{OP}.$$

Example: If the same point on the rigid body is translated from P to P′′, were P′′-P = $\boldsymbol{\delta}$ is the same for all points of the body, then

$$\mathbf{OP}'' = \mathbf{OP} + \delta \quad \Leftrightarrow \quad \mathbf{PP}' = \delta$$

If the body is first rotated and then translated the displacement of P is

$$\mathbf{PP}' = \delta + \left(\boldsymbol{q} \bullet \mathbf{OP} \bullet \boldsymbol{q}^{-1}\right) - \mathbf{OP}\ .$$

while if the body is translated first and then rotated, then the displacement is

$$\mathbf{PP}' = \left(\boldsymbol{q} \bullet \left(\mathbf{OP} + \delta\right) \bullet \boldsymbol{q}^{-1}\right) - \mathbf{OP}\ .$$