Notes on Shear

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Shear is a displacement of parts of a material matrix relative to other parts. If we take a small rectangular slice of the material, then the rectangle in the unstrained material becomes a parallelogram in the strained material. For most solid materials, the displacement is small, but in fluids it may be considerable. When shear is diagramed, it is usually represented as a horizontal displacement, but it may also be seen as a combination of a rotation and an elongation.

We will start with a level of analysis that will not seem immediately related, but which will ultimately lead to a mathematical method for representing shear. We start with the representation of rotations of orthogonal coordinates in a plane.

Rotation in the plane of the basis vectors

If we have two orthogonal axes in a plane, which will be taken for simplicity to be the **i,j**-plane, then we can represent the rotation of the coordinates about their common origin by quaternion multiplication. Let a vector in the original coordinate system be represented by **r** = {0, x**i**, y**j**, 0**k**} and the rotation be through an angle θ . Then, the new coordinates, $\mathbf{r}'$, are given by the following expression.

$$\mathbf{r}' = \boldsymbol{R} * \mathbf{r} ;$$
$$\boldsymbol{r} = \mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} ,$$
$$\boldsymbol{R} = \cos\theta + \sin\theta * \mathbf{k}$$

However, in order to make generalization more convenient, by not assuming that the rotation is about an orthogonal axis, this may also be written as follows.

$$\mathbf{r}' = \boldsymbol{R} * \mathbf{r} * \boldsymbol{R}^{-1} ;$$
$$\mathbf{r} = \mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} ,$$
$$\boldsymbol{R} = \cos\frac{\theta}{2} + \boldsymbol{k}\sin\frac{\theta}{2} .$$

Rotation about an axis perpendicular to the plane of the basis vectors leaves the basis vectors oriented in the same way relative to each other. Any orthonormal transformation can be expressed as a combination of rotations and translations.

Projections of rotated basis vectors

What happens if the axis of rotation is not perpendicular to the plane of the basis vectors? In particular, what happens if it is in the plane of the basis vectors? We choose particular axes because they yield simpler computation, but they are not fundamentally different from other axes for the features that we are going to examine.

Converging axes: Suppose that the axis of rotation is at a 45° angle to both axes, a diagonal that passes midway between the **x** and **y** axes. One can easily visualize this rotation. One axis rises above the plane and one sinks below the plane. If we look at the projection of the axes upon the plane, then it appears that the two axes move towards each other in much the same way as the sides of the parallelogram converge during shear. More particularly, the portions of the axes with the same sign converge and the portions with different signs diverge. We can express this transformation as follows.

$$\begin{aligned}\boldsymbol{R} &= \cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\left(\mathbf{i}\cos\frac{\pi}{4} + \mathbf{j}\sin\frac{\pi}{4}\right)\\ &= \cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\left(\frac{\mathbf{i}}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}}\right)\end{aligned}$$

The axis of rotation is at a 45° angle to both basis vectors and the axes rotate through an angle γ.

$$\mathbf{r}' = \boldsymbol{R} * \mathbf{r} * \boldsymbol{R}^{-1}$$

If we set **r** equal to each of the basis vectors, then the new clipped basis vectors in the **i,j**-plane are the projections of the full basis vectors upon the **i,j**-plane.

Diverging axes: If the axis of rotation is orthogonal to this diagonal and in the plane of the basis vectors, then it will form the other diagonal between the basis vectors. Rotation about this diagonal will cause the positive basis vectors to rise above or sink below the plane of the basis vectors together. The projections of the basis vectors on the horizontal plane will diverge until they reach 180°, when the rotation reaches 90°. The rotation quaternion for this transformation will be as follows.

$$\begin{aligned}\boldsymbol{R} &= \cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\left(\boldsymbol{i}\cos\frac{\pi}{4} - \boldsymbol{j}\sin\frac{\pi}{4}\right)\\ &= \cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\left(\frac{\boldsymbol{i}}{\sqrt{2}} - \frac{\boldsymbol{j}}{\sqrt{2}}\right).\end{aligned}$$

Note that either of these will work to model shear. The second is equivalent to the first rotated through 90°.

Generalization of the calculation

The calculation can be generalized to make it more broadly applicable. First, the initial basis vectors are not necessarily orthogonal. If we want the two axes to transform symmetrically then we have to find the diagonal that is midway between the two basis vectors. The ratio of the **y** axis to the **x** axis ($\rho_{\mathbf{y:x}}$) gives the orientation of the plane that contains the two vectors, in terms of the perpendicular to that plane (ν), and the angle between them, in terms of the angle of the quaternion (α_ρ). The vector of the diagonal ($\mathbf{V}(\mathbf{d})$) is the product of the ratio quaternion with half the angle and the **x** axis.

$$\rho_{\mathbf{y:x}}(\alpha_\rho,\nu)=\frac{\mathbf{y}}{\mathbf{x}};$$
$$\alpha_\rho=\angle(\rho),\nu=\mathbf{V}(\rho);$$
$$\mathbf{V}(\mathbf{d})=\vec{\boldsymbol{d}}=\rho_{\mathbf{y:x}}\left(\frac{\alpha_\rho}{2},\nu\right)*\mathbf{x}.$$

The initial basis vectors can then be rotated about the diagonal by multiplying them by the rotation quaternion with the diagonal as its vector and an angle that is half of the desired angular excursion.

$$\begin{bmatrix}\mathbf{x}'\\ \mathbf{y}'\end{bmatrix}=\boldsymbol{R}*\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}*\boldsymbol{R}^{-1};$$
$$\boldsymbol{R}(\gamma,\vec{\mathbf{d}})=\cos\frac{\gamma}{2}+\vec{\mathbf{d}}*\sin\frac{\gamma}{2}.$$

It remains to be determined what the angular excursion, γ, should be to produce the appropriate projection image. This will be considered below.

The next step is to compute the projection of the rotated basis vectors back upon the original plane of the basis vectors. This may be accomplished by noting that the vector of the ratio of the original basis vectors, $\vec{\mathbf{d}}$, is a perpendicular to the projection plane. The projection lies in the plane that contains the ratio vector and the transformed basis vector. That plane is determined by taking the ratio of those vectors, then rotating $\vec{\mathbf{d}}$ though $\pi/2$ radians in the direction of the transformed basis vector.

$$\rho_{\mathbf{d}:\mathbf{x}'}\left(\alpha_{\rho(\mathbf{d},\mathbf{x}')},\nu_{\rho(\mathbf{d},\mathbf{x}')}\right) = \frac{\mathbf{x}'}{\mathbf{V}\left(\rho_{\mathbf{y}:\mathbf{x}}\right)}, \quad \rho_{\mathbf{d}:\mathbf{y}'}\left(\alpha_{\rho(\mathbf{d},\mathbf{y}')},\nu_{\rho(\mathbf{d},\mathbf{y}')}\right) = \frac{\mathbf{y}'}{\mathbf{V}\left(\rho_{\mathbf{y}:\mathbf{x}}\right)};$$

$$\alpha_{\rho(\mathbf{d},\mathbf{x}')} = \angle\left(\rho_{\mathbf{d}:\mathbf{x}'}\right), \nu_{\rho(\mathbf{d},\mathbf{x}')} = \mathbf{V}\left(\rho_{\mathbf{d}:\mathbf{x}'}\right),$$

$$\alpha_{\rho(\mathbf{d},\mathbf{y}')} = \angle\left(\rho_{\mathbf{d}:\mathbf{y}'}\right), \nu_{\rho(\mathbf{d},\mathbf{y}')} = \mathbf{V}\left(\rho_{\mathbf{d}:\mathbf{y}'}\right);$$

$$\mathbf{x}'_{\mathbf{projection}} = \hat{\mathbf{x}} = \rho_{\mathbf{d}:\mathbf{x}'}\left(\alpha = \frac{\pi}{2}, \nu_{\rho(\mathbf{d},\mathbf{x}')}\right) * \vec{\mathbf{d}},$$

$$\mathbf{y}'_{\mathbf{projection}} = \hat{\mathbf{y}} = \rho_{\mathbf{d}:\mathbf{y}'}\left(\alpha = \frac{\pi}{2}, \nu_{\rho(\mathbf{d},\mathbf{y}')}\right) * \vec{\mathbf{d}}.$$

A side benefit of this approach is that the projected image of a rotated basis vector has a unit length, because $\vec{\mathbf{d}}$ has a unit length, so it is not necessary to specifically re-scale.

Now, the projection in the plane of the original basis vectors has been computed, but it is rotated in that plane relative to the original basis vectors, therefore, it is necessary to re-align it with the original **x** axis. This is done by computing the ratio of the original **x** axis, **x**, to the projection of the original **x** axis, $\hat{\mathbf{x}}$, then multiplying $\hat{\mathbf{x}}$ by the ratio. This realigns the $\hat{\mathbf{x}}$ axis with the **x** axis, then we multiply the **y** axis by the same ratio to rotate it equivalently.

$$\hat{\mathbf{x}}_0 = \rho\left(\frac{\mathbf{x}}{\hat{\mathbf{x}}}\right) * \hat{\mathbf{x}}, \quad \hat{\mathbf{y}}_0 = \rho\left(\frac{\mathbf{x}}{\hat{\mathbf{x}}}\right) * \hat{\mathbf{y}}$$

If the shear also rotates then the sheared vectors may be aligned with some axis other than the x axis. The principle is the same, except that we replace x by the other axis in the last equations.

This approach is computationally intense, but it is also generally applicable. It basically says that we can take any two basis vectors in space and compute the consequences of a particular shear upon their configuration. We can also take any point in the plane that they define and compute how it will be transformed by the shear.

$$\mathbf{r}' = \boldsymbol{R} * \mathbf{r} * \boldsymbol{R}^{-1};$$

$$\rho_{\mathbf{d}:\mathbf{r}'} = \frac{\mathbf{r}'}{\rho_{\mathbf{y}:\mathbf{x}}};$$

$$\hat{\mathbf{r}} = \rho_{\mathbf{d}:\mathbf{r}'}\left(\frac{\pi}{2}, \nu_{\rho(\mathbf{d},\mathbf{r}')}\right) * \vec{\mathbf{d}} * |\mathbf{r}|;$$

$$\hat{\mathbf{r}}_0 = \rho\left(\frac{\mathbf{x}}{\hat{\mathbf{x}}}\right) * \hat{\mathbf{r}}.$$

We are left with the problem of determining the amount of rotation of the basis vectors that will give the required amount of shear. Let this rotation of the basis vectors about the diagonal be called the torsion and let it be symbolized by γ.

If we are starting with a divergence of the basis vectors. Then the analysis is almost the same, we just start with $\rho_{\mathbf{y:x}} = \frac{-\mathbf{y}}{\mathbf{x}}$.

Finding the torsion that gives the required shear: contraction

The torsion for a given amount of shear can be computed based upon the following analysis. The objects that are discussed are illustrated in the figure. The thick circle that forms the boundary of the plot is the ring that is traced out by the axes as they swing about the axis of rotation. If we are considering orthogonal unitary basis vectors then it is a ring with a diameter of $\sqrt{2}$ lying a distance of $\frac{\sqrt{2}}{2}$ away from the origin. In the figure, we are viewing the ring and origin from a point in the upper right quadrant. The point **O** is the origin and the yellow triangle is the **i,j**-plane between the two axes. The green triangle is the same triangle rotated through an angle γ. The blue triangle is the image of the upper half of the rotated plane in the horizontal plane. The red triangle is the angle between the rotated basis vectors and their image in the horizontal plane. The angle α is half the angle between the two sheared axes. It is a given in the present situation, because it is the measure of the amount of shear. We wish to know γ, the torsion of the basis vectors that causes their image in the horizontal plane to have the appropriate amount of shear.

We start with the observation that the basis vectors are unitary vectors, therefore have a length of 1.0. We consider the part of the rotated basis vector's triangle that lies above the image plane. The outer edge of the triangles, most distant from the origin, is a distance of $\mathbf{x} = \cos\frac{\theta}{2}$, where θ is the angle between the basis vectors. In this case $\theta = \frac{\pi}{4}$ radians, therefore $\mathbf{x} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \approx 0.707$. We know that the outer edge of the upper green triangle is $\mu = \sin\frac{\theta}{2}$. In the present case, $\mu = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. The junction between the horizontal and vertical triangles, λ, is given by the expression $\lambda = \frac{\mathbf{x}}{\cos\alpha}$, where α is half the angle between the sheared basis vectors. We can compute **y** by a number of means.

$$\mathbf{y} = \lambda\sin\alpha = \sqrt{\lambda^2 - \mathbf{x}^2}$$

Given **y** and μ, we can compute the torsion, γ.

$$\gamma = \cos^{-1}\frac{\mathbf{y}}{\mu}.$$

By algebraic manipulation, we can determine the value of γ, given α and θ.

$$\gamma = \cos^{-1}\left(\frac{\sqrt{\frac{1}{\cos^2\alpha} - 1}}{\tan\frac{\theta}{2}}\right).$$

When the basis vectors are orthogonal, then the tangent of 45° is unity and the expression looks less formidable and it depends only on the amount of shear, α.

$$\gamma = \cos^{-1}\left(\sqrt{\frac{1}{\cos^2\alpha} - 1}\right).$$

At this point we have all the necessary information to compute the distortion introduced by a shear of a particular size. The next step is to compute the shear, given a particular displacement of the **y** axis.

Finding the torsion that gives the required shear: dilation

The situation is simpler for the case of dilation, when the angle between the basis vectors increases. The axis of rotation is the diagonal that passes midway between the positive **x** axis and the negative **y** axis. Therefore. the wedge between the axes rises uniformly above the **i,j**-plane or sinks uniformly beneath it. If the angle between the basis vectors prior to the transformation is θ, then the angle is the same for the rotated axes. The rotation of the basis vectors is through an angle γ and the angle between the basis vectors in the projection image in the **i,j**-plane is α. The problem that we are dealing with at this point is to determine the magnitude of γ necessary to produce a shear of α. The length of the edges of the untransformed basis vectors and the rotated basis vectors is 1.0 and the outer edge is 2 sin θ. If we symbolize the midline of the image wedge as **x** and the outer margin of the half wedge as **y**, then we can write down their magnitudes by inspection, because they are the projections of the rotated wedge.

$$\mathbf{y} = \sin\theta \text{ and } \mathbf{x} = \cos\gamma * \cos\theta .$$

We can write down an expression for the angle α, which allows us to determine the relationship between the amount of rotation of the basis vectors and the angle of the basis vectors in the image.

$$\tan\alpha = \frac{\mathbf{y}}{\mathbf{x}} = \frac{\sin\theta}{\cos\gamma\cos\theta};$$

$$\cos\gamma = \frac{\tan\theta}{\tan\alpha};$$

$$\gamma = \cos^{-1}\left(\frac{\tan\theta}{\tan\alpha}\right).$$

When the original basis vectors are orthogonal then the expression becomes

$$\gamma = \cos^{-1}\left(\frac{1}{\tan\alpha}\right) = \cos^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right).$$

Note that this expression has real solutions only for $\alpha > 45°$, which is where we are applying it.

Displacements lead to shears

If we visualize a square in the matrix material prior to a strain, where we pick the square to be oriented so that the stress force pushes perpendicular to the **y** axis, then the displacement of the y axis one unit from the origin is δ. The angular shift of the **y** axis is ϕ_δ.

$$\phi_\delta = \sin^{-1}\frac{\delta}{1.0} = \sin^{-1}\delta$$

This shift makes the inner angle between the **x** and **y** axes, θ, less.

$$2\alpha = \frac{\pi}{2} - \phi_\delta$$

Thus, α is the shear and we can use the work up to this point to analyze the effect of the displacement upon points in the matrix that lie in the vicinity of the square, assuming that the distortion of the material is nearly linear over small distances.

Multidimensional shear

Many problems can be abstracted to a strain in a plane, but some cannot. We now consider some of the latter.

We start with a set of orthogonal basis vectors {**x, y, z**} which are distorted into a skewed set of basis vectors {**x', y', z'**}. There is a simple relationship between the original basis vectors. The ratio of any two basis vectors is the third basis vector or its negative. This is not true of the skewed set of basis vectors. However, there is a perfectly good ratio for each pair of basis vectors.

$$\rho'_{\mathbf{y:x}} = \frac{\mathbf{y}'}{\mathbf{x}'}; \quad \rho'_{\mathbf{z:x}} = \frac{\mathbf{z}'}{\mathbf{x}'}; \quad \textit{etc}.$$

We cannot divide the skewed basis vectors by the orthogonal set in the usual manner, because the procedure would not give a correct result. The usual definition of the ratio of frames of reference assumes that the two frames that are being divided are both orthogonal sets of vectors. We can still use the ratio of frames of reference, but it is necessary to attach an orthogonal frame of reference to each, to provide an orientation to one of the basis vectors. By default the first component of the frame of reference will be aligned with the **x** axis. Unless, set otherwise, it is assumed that the frame for the orthogonal set of basis vectors is aligned with the basis vectors. After the transformation that gives rise to the skewed basis, the frame is still orthogonal and it gives an orientation to the skewed vectors. If $\boldsymbol{f}_0$ is the frame of reference for the orthogonal set and $\boldsymbol{f}_1$ is the frame for the skewed set, then –

$$\rho_{\mathbf{1:0}} = \frac{\boldsymbol{f}_1}{\boldsymbol{f}_0} \quad \Leftrightarrow \quad \boldsymbol{f}_1 = \rho_{\mathbf{1:0}} * \boldsymbol{f}_0 * \rho_{\mathbf{1:0}}^{-1}$$

We can write the ratios for the skewed axes as follows.

$$\rho_{\mathbf{x}':\mathbf{x}_1} = \frac{\mathbf{x}'}{\mathbf{x}_1}, \quad \rho_{\mathbf{y}':\mathbf{y}_1} = \frac{\mathbf{y}'}{\mathbf{y}_1}, \quad \rho_{\mathbf{z}':\mathbf{z}_1} = \frac{\mathbf{z}'}{\mathbf{z}_1}$$

If we have a vector in the orthogonal space that is expressed by the following equation,

$$\mathrm{v} = \mathrm{a}\mathbf{x}_0 + \mathrm{b}\mathbf{y}_0 + \mathrm{c}\mathbf{z}_0 ,$$

and the basis vectors are given by the following expressions,

$$\mathbf{x}_0 = \alpha_x \mathbf{i} + \beta_x \mathbf{j} + \gamma_x \mathbf{k} ,$$
$$\mathbf{y}_0 = \alpha_y \mathbf{i} + \beta_y \mathbf{j} + \gamma_y \mathbf{k} ,$$
$$\mathbf{z}_0 = \alpha_z \mathbf{i} + \beta_z \mathbf{j} + \gamma_z \mathbf{k} ,$$

then

$$\mathbf{x}_1 = \rho_{\mathbf{x}':\mathbf{x}_1} * \rho_{\mathbf{1:0}} * \mathbf{x}_0 * \rho_{\mathbf{1:0}}^{-1} * \rho_{\mathbf{x}':\mathbf{x}_1}^{-1} = \rho_{\mathbf{x}':\mathbf{x}_1} * \rho_{\mathbf{1:0}} * \left(\alpha_x \mathbf{i} + \beta_x \mathbf{j} + \gamma_x \mathbf{k}\right) * \rho_{\mathbf{1:0}}^{-1} * \rho_{\mathbf{x}':\mathbf{x}_1}^{-1} ;$$
$$\mathbf{y}_1 = \rho_{\mathbf{y}':\mathbf{y}_1} * \rho_{\mathbf{1:0}} * \boldsymbol{y}_0 * \rho_{\mathbf{1:0}}^{-1} * \rho_{\mathbf{y}':\mathbf{y}_1}^{-1} = \rho_{\mathbf{y}':\mathbf{y}_1} * \rho_{\mathbf{1:0}} * \left(\alpha_y \mathbf{i} + \beta_y \mathbf{j} + \gamma_y \mathbf{k}\right) * \rho_{\mathbf{1:0}}^{-1} * \mathbf{r}_{\mathbf{y}':\mathbf{y}_1}^{-1} ;$$
$$\mathbf{z}_1 = \rho_{\mathbf{z}':\mathbf{z}_1} * \rho_{\mathbf{1:0}} * \boldsymbol{z}_0 * \rho_{\mathbf{1:0}}^{-1} * \rho_{\mathbf{z}':\mathbf{z}_1}^{-1} = \rho_{\mathbf{z}':\mathbf{z}_1} * \rho_{\mathbf{1:0}} * \left(\alpha_z \mathbf{i} + \beta_z \mathbf{j} + \gamma_z \mathbf{k}\right) * \rho_{\mathbf{1:0}}^{-1} * \rho_{\mathbf{z}':\mathbf{z}_1}^{-1} .$$

and

$$\nu = a\mathbf{x}_1 + b\mathbf{y}_1 + c\mathbf{z}_1 \ .$$

Representing Deformation

If a force is impressed upon a small region of a malleable material, then it will cause the material to deform. If we imagine a small cube of the material, then the cube is deformed by the shifting of points about a central point. If there are axes passing through the point in a manner that codifies the three-dimensional configuration of the material, then the shifting of the axes encodes the deformation. However, the deformation may be in any direction. The effect of the deformation will be the assumed to be applied to a rigid armature that is aligned with each axis. If **w** is an axis and **d** is the deformation, then $\mathbf{w}'$ will be the deformed axis.

$$\mathbf{w}' = \mathbf{w} + \mathbf{d}$$

From a set of deformed axes one can compute the change in volume by computing the volume prior to the deformation and after it occurs, then computing the difference. If there is a change in volume, then there must be a compaction or dispersion of the matrix of the material. These changes may be a source of a distorting force.

Oriented filaments

A filament follows a trajectory, straight or curved. This is one spatial attribute of the filament. It will be argued here that the filament may also be oriented. To be oriented, it must have a direction and a normal and a perpendicular as per differential geometry.